

**VERTEX-ALGEBRAIC STRUCTURE OF PRINCIPAL SUBSPACES OF THE
BASIC MODULES FOR TWISTED AFFINE KAC-MOODY LIE ALGEBRAS OF
TYPE $A_{2n-1}^{(2)}, D_n^{(2)}, E_6^{(2)}$**

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ABSTRACT. We obtain a presentation of principal subspaces of the basic modules for the twisted affine Kac-Moody Lie algebras of type $A_{2n-1}^{(2)}, D_n^{(2)}$ and $E_6^{(2)}$. Using this presentation, we construct exact sequences among these principal subspaces, and use these exact sequences to obtain recursions satisfied by graded dimensions of the principal subspaces. Solving these recursions, we obtain the graded dimensions of the principal subspaces.

1. INTRODUCTION

Principal subspaces of standard modules for untwisted affine Lie algebras were introduced and studied by Feigin and Stoyanovsky in [FS1]–[FS2]. Motivated by work of Lepowsky and Primc [LP], Feigin and Stoyanovsky noticed difference-two type partition conditions in these structures. In particular, Feigin and Stoyanovsky showed that the multigraded dimensions of the principal subspace of the basic modules for $A_1^{(1)}$ are the sum-sides of the Rogers-Ramanujan identities, and more generally, the multigraded dimensions of the principal subspaces of the higher level standard modules for $A_1^{(1)}$ are the sum-sides of the Andrews-Gordon identities.

In [CLM1]–[CLM2], Capparelli, Lepowsky, and Milas, studied the principal subspaces of the standard modules for $A_1^{(1)}$ using the theory of vertex operator algebras and intertwining operators. Under certain assumptions (namely, presentations of principal subspaces using generators and relations used in [FS1]–[FS2]), they constructed exact sequences among these principal subspaces. They used these exact sequences to show that the multigraded dimensions of the principal spaces satisfy the Rogers-Ramanujan and Rogers-Selberg recursions. From these recursions, they recovered the graded dimensions found in [FS1]–[FS2]. In [CaLM1]–[CaLM2], Calinescu, Lepowsky, and Milas, again using the theory of vertex operator algebras and intertwining operators, proved the presentations used in [FS1]–[FS2] and [CLM1]–[CLM2]. In [CaLM3], they extended their work to study the principal subspaces of basic modules for untwisted affine Lie algebras of type A, D, E . They also constructed exact sequences among the principal subspaces, found the recursions that the multigraded dimensions satisfy, and solved those recursions to find the multigraded dimensions of the principal subspaces.

Principal subspaces have been studied from a vertex-algebraic point of view in many other works. We give a brief overview of recent work in this area. In [C1] and [S1], principal subspaces of higher level standard $A_2^{(1)}$ -modules have been studied in a spirit similar to [CLM1]–[CLM2] and [CaLM1]–[CaLM2]. These principal subspaces were also studied in [AKS], [FFJMM], and in other works by these authors. Georgiev used the theory of vertex operator algebras and intertwining operators to construct combinatorial bases for principal subspaces of certain higher level standard modules for untwisted affine Lie algebras of type A, D, E in [G], and this work has since been extended to untwisted affine Lie algebras of type B, C by Butorac in [Bu1]–[Bu2] and to the quantum group case by Kožić in [Ko]. Principal subspaces of standard modules for more general lattice vertex operator algebras have been studied in [MPe], [P], and [Ka1]–[Ka2]. Certain commutative variants of principal

subspaces, called Feigin-Stoyanovsky type subspaces, have been studied in [Ba], [J1]–[J2], [JP], and [T1]–[T3]. The multigraded dimensions of principal subspaces are Nahm sums (cf. [Za]) or closely resemble Nahm sums, and modularity properties of these sums have been studied in [BCFK] for the untwisted affine Lie algebras of type A and in [CalMPe] for the twisted affine Lie algebra $A_4^{(2)}$.

The present work is an extension of the work on principal subspaces of basic modules for twisted affine Lie algebras initiated in [CalLM4] and continued in [CalMPe] and [PS]. We study the principal subspaces of the basic modules for twisted affine Lie algebras of type $A_{2n-1}^{(2)}$ for $n \geq 2$, $D_n^{(2)}$ for $n \geq 4$, and $E_6^{(2)}$. We prove a presentation of these principal subspaces and use this presentation to find their multigraded dimensions. It is important to note that the ideas involved in this work and in [CalLM4], [CalMPe], and [PS] all require slightly different constructions and technical lemmas, and thus have been studied on a case-by-case basis in these works. To date, there is no known way to handle all the cases simultaneously. We also note here that in this paper, as well as in [PS], the multigraded dimensions of the principal subspaces closely resemble Nahm sums, but are not Nahm sums themselves due to the form that the denominator of the summands (namely, the appearance of terms of the form $(q^2; q^2)_n$ in the denominators).

We now give a brief outline of this paper. In Section 2 of, we follow [L1] and [CalLM4] and give the vertex-algebraic construction of the basic modules V_L^T for the twisted affine Lie algebras $A_{2n-1}^{(2)}$, $D_n^{(2)}$, and $E_6^{(2)}$, viewed as twisted modules for the lattice vertex operator algebras V_L of type $A_{2n-1}^{(1)}$, $D_n^{(1)}$, and $E_6^{(1)}$. This construction uses the Dynkin diagram automorphisms for the finite dimensional Lie algebras of type A, D, E , which we denote ν . In Section 3, we recall the notion of principal subspace (cf. [FS1]–[FS2], [CLM1]–[CLM2], [CalLM1]–[CalLM4], and many others) and define the principal subspaces W_L^T of the basic modules constructed in Section 2. We also define certain natural operators which annihilate the highest weight vectors of the basic modules, as well as left ideals generated by these operators. In Section 4, we define certain natural “shifting” maps among these ideals, and prove various containment properties of these maps. In Section 5, we use these maps to show that the left ideals defined in Section 3 give presentations of our principal subspaces. We use these presentations to construct exact sequences among our principal subspaces, find recursions satisfied by their multigraded dimensions:

$$(1.1) \quad \chi'(\mathbf{x}; q) = \chi'(x_1, \dots, x_{i-1}, q^2 x_i, x_{i+1}, \dots, x_k; q) + x_i q^2 \chi'(q^{2\langle(\alpha_{i_1})_{(0)}(\alpha_i)_{(0)}\rangle} x_1, \dots, q^{2\langle(\alpha_{i_k})_{(0)}(\alpha_i)_{(0)}\rangle} x_k; q)$$

if $\nu\alpha_i = \alpha_i$ and

$$(1.2) \quad \chi'(\mathbf{x}; q) = \chi'(x_1, \dots, x_{i-1}, qx_i, x_{i+1}, \dots, x_k; q) + x_i q \chi'(q^{2\langle(\alpha_{i_1})_{(0)}(\alpha_i)_{(0)}\rangle} x_1, \dots, q^{2\langle(\alpha_{i_k})_{(0)}(\alpha_i)_{(0)}\rangle} x_k; q)$$

if $\nu\alpha_i \neq \alpha_i$. Here, in each case, we denote the simple roots by α_i , and denote by $(\alpha_i)_{(0)}$ the projection of the root α_i onto the 0 eigenspace of ν extended to the Cartan subalgebra of the underlying finite dimensional Lie algebra. Consequently, solving these recursions, we obtain the multigraded dimensions of the principal subspaces:

$$(1.3) \quad \chi'(\mathbf{x}; q) = \sum_{\mathbf{m} \in (\mathbb{Z}_{\geq 0}^k)} \frac{q^{\frac{\mathbf{m}^t A \mathbf{m}}{2}}}{(q^{a_1}; q^{a_1})_{m_1} \cdots (q^{a_k}; q^{a_k})_{m_k}} x_1^{m_1} \cdots x_k^{m_k}$$

where $a_j = 2$ if $\nu\alpha_{i_j} = \alpha_{i_j}$, $a_j = 1$ if $\nu\alpha_{i_j} \neq \alpha_{i_j}$ and $A = 2(\langle(\alpha_{i_1})_{(0)}, (\alpha_{i_k})_{(0)}\rangle)$. The ideas in Sections 3, 4, and 5 are natural analogues of ideas used in [CalLM1]–[CalLM4] and [PS] to prove similar presentations, adapted to the setting of this paper.

2. THE SETTING

Let

$$(2.1) \quad L = \mathbb{Z}\alpha_1 \oplus \cdots \mathbb{Z}\alpha_l$$

be an integral lattice with associated symmetric non-degenerate bilinear form $\langle \cdot, \cdot \rangle : L \times L \rightarrow \mathbb{Z}$ such that $\langle \alpha, \alpha \rangle \in 2\mathbb{Z}$ for all $\alpha \in L$. In this setting we often say L is an even lattice. We also assume that L admits an isometry $\nu : L \rightarrow L$ such that

$$(2.2) \quad \nu^k = 1$$

and

$$(2.3) \quad \left\langle \nu^{\frac{k}{2}} \alpha, \alpha \right\rangle \in 2\mathbb{Z} \quad \text{for all } \alpha \in L$$

if k is even. In the language of [L1], [CalLM4] and others, we take $k = 2$, and in addition assume that for all $\alpha \in L$ we have $\langle \nu^{k/2} \alpha, \alpha \rangle \in 2\mathbb{Z}$.

Remark 2.1. The root lattices of the Lie algebras A_{2n-1} , D_n , E_6 are special cases of this general set-up. We will exploit this to analyze the principal subspaces of the level one standard modules of $A_{2n-1}^{(2)}$, $D_n^{(2)}$ and $E_6^{(2)}$. The $A_{2n}^{(2)}$ case was examined in [CalMPe], and required $k = 4$ as the period of ν (which had order 2), doubling the order of the Dynkin diagram automorphism for the Lie algebra A_{2n} .

Following Section 2 of [CalLM4] (also [L1]) while specializing to our setting, we fix $\eta = -1$, the primitive second root of unity, and set $\eta_0 = (-1)^2 \eta = -1$. Define the functions $C_0 : L \times L \rightarrow \mathbb{C}^\times$ and $C : L \times L \rightarrow \mathbb{C}^\times$ by

$$(2.4) \quad C_0(\alpha, \beta) = (-1)^{\langle \alpha, \beta \rangle}$$

and

$$(2.5) \quad C(\alpha, \beta) = \prod_{j=0}^1 (-1)^j \langle \nu^j \alpha, \beta \rangle = (-1)^{\langle \alpha, \beta \rangle}.$$

Here we have $C = C_0$.

We define the central extension \hat{L} of L by $\langle -1 \rangle$ using the commutator map $C_0 (= C)$. In other words, we have an exact sequence

$$(2.6) \quad 1 \longrightarrow \langle -1 \rangle \longrightarrow \hat{L} \xrightarrow{\pi} L \longrightarrow 1$$

so that $aba^{-1}b^{-1} = C_0(\bar{a}, \bar{b})$ for $a, b \in \hat{L}$.

Remark 2.2. In previous related work ([CalLM4], [CalMPe], and [PS]), C_0 and C were inequivalent maps and thus determined two inequivalent central extensions \hat{L} and \hat{L}_ν , respectively.

After [L1] and [CalLM4], we let

$$\begin{aligned} e : L &\rightarrow \hat{L} \\ \alpha &\mapsto e_\alpha \end{aligned}$$

be a normalized section of \hat{L} so that

$$(2.7) \quad e_0 = 1$$

and

$$(2.8) \quad \overline{e_\alpha} = \alpha \quad \text{for all } \alpha \in L.$$

We define $\epsilon_{C_0} : L \times L \rightarrow \langle -1 \rangle$ by the condition

$$(2.9) \quad e_\alpha e_\beta = \epsilon_{C_0}(\alpha, \beta) e_{\alpha+\beta} \quad \text{for all } \alpha, \beta \in L.$$

With this set-up $\epsilon_{C_0}(= \epsilon_C)$ is a normalized 2-cocycle associated to the commutator map $C_0(= C)$. In other words,

$$\begin{aligned} \epsilon_{C_0}(\alpha, \beta) \epsilon_{C_0}(\alpha + \beta, \gamma) &= \epsilon_{C_0}(\beta, \gamma) \epsilon_{C_0}(\alpha, \beta + \gamma) \\ \epsilon_{C_0}(0, 0) &= 1 \end{aligned}$$

and

$$(2.10) \quad \frac{\epsilon_{C_0}(\alpha, \beta)}{\epsilon_{C_0}(\beta, \alpha)} = C_0(\alpha, \beta).$$

We choose our 2-cocycle to be the \mathbb{Z} -bilinear map determined by

$$(2.11) \quad \epsilon_{C_0}(\alpha_i, \alpha_j) = \begin{cases} 1 & \text{for } i \leq j \\ (-1)^{\langle \alpha_i, \alpha_j \rangle} & \text{for } i \geq j. \end{cases}$$

As in [L1] and [CalLM4], we lift the isometry ν of L to an automorphism $\hat{\nu}$ of \hat{L} such that

$$(2.12) \quad \overline{\hat{\nu}a} = \nu \overline{a} \quad \text{for } a \in \hat{L}.$$

and choose $\hat{\nu}$ so that

$$(2.13) \quad \hat{\nu}a = a \quad \text{if } \nu \overline{a} = \overline{a},$$

and thus $\hat{\nu}^2 = 1$. Set

$$(2.14) \quad \hat{\nu}e_\alpha = \psi(\alpha) e_{\nu\alpha}$$

where $\psi : L \rightarrow \langle -1 \rangle$ is defined by

$$(2.15) \quad \psi(\alpha) = \begin{cases} \epsilon_{C_0}(\alpha, \alpha) & \text{if } L \text{ is type } A_{2n-1} \\ 1 & \text{if } L \text{ is type } D_n \\ (-1)^{r_3 r_6} \epsilon_{C_0}(\alpha, \alpha) & \text{if } L \text{ is type } E_6 \text{ and } \alpha = \sum_{i=1}^6 r_i \alpha_i. \end{cases}$$

Checking that $\hat{\nu}(e_\alpha e_\beta) = \hat{\nu}e_\alpha \hat{\nu}e_\beta$ is clear given that

$$(2.16) \quad \epsilon_{C_0}(\nu\alpha, \nu\beta) = \begin{cases} \epsilon_{C_0}(\beta, \alpha) & \text{if } L \text{ is type } A_{2n-1} \\ \epsilon_{C_0}(\alpha, \beta) & \text{if } L \text{ is type } D_n \\ (-1)^{r_6 s_3 + r_3 s_6} \epsilon_{C_0}(\beta, \alpha) & \text{if } L \text{ is type } E_6, \alpha = \sum_{i=1}^6 r_i \alpha_i, \text{ and } \beta = \sum_{i=1}^6 s_i \alpha_i. \end{cases}$$

Observe that this choice of lifting satisfies (2.12), (2.13), and

$$(2.17) \quad \hat{\nu}e_{\alpha_i} = e_{\nu\alpha_i},$$

where α_i is a simple root.

Now we recall the construction of the vertex operator algebra V_L while lifting $\hat{\nu}$ to an automorphism of V_L which we also denote by $\hat{\nu}$, and use the space \hat{L}_ν to construct $\hat{\nu}$ -twisted modules of V_L . This closely follows [L1] and [FLM2]. We begin by setting

$$(2.18) \quad \mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$$

and extend the bilinear form on L to a \mathbb{C} -bilinear form on \mathfrak{h} . We view \mathfrak{h} as an abelian Lie algebra and consider its affinization

$$(2.19) \quad \hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}k$$

where

$$(2.20) \quad [\alpha \otimes t^r, \beta \otimes t^s] = \langle \alpha, \beta \rangle r \delta_{r+s,0} k$$

for $\alpha, \beta \in \mathfrak{h}$ and $r, s \in \mathbb{Z}$, where k is central. We also endow $\hat{\mathfrak{h}}$ with a natural \mathbb{Z} -grading given by $\text{wt}(\alpha \otimes t^m) = -m$ and $\text{wt}(k) = 0$ for $\alpha \in \mathfrak{h}$ and $m \in \mathbb{Z}$. Now form the module

$$(2.21) \quad M(1) = U(\hat{\mathfrak{h}}) \otimes_{U(\mathfrak{h} \otimes \mathbb{C}[t] \oplus \mathbb{C}k)} \mathbb{C} \cong S(\hat{\mathfrak{h}}^-) \text{ linearly}$$

where $\hat{\mathfrak{h}}^- = \mathfrak{h} \otimes t^{-1} \mathbb{C}[t^{-1}]$. Observe that $M(1)$ inherits the \mathbb{Z} -grading from $\hat{\mathfrak{h}}$.

Next we form the \hat{L} -module

$$(2.22) \quad \mathbb{C}\{L\} = \mathbb{C}[\hat{L}] \otimes_{\mathbb{C}[\{-1\}]} \mathbb{C} \cong \mathbb{C}[L] \text{ (linearly)}.$$

Using the inclusion $\iota : \hat{L} \rightarrow \mathbb{C}\{L\}$ we write $\iota(a) = a \otimes 1$. This module is \mathbb{C} -graded so that

$$(2.23) \quad \text{wt}(\iota(1)) = 0$$

and

$$(2.24) \quad \text{wt}(e_\alpha) = \frac{1}{2} \langle \alpha, \alpha \rangle,$$

for $\alpha \in L$. We set

$$(2.25) \quad \begin{aligned} V_L &= M(1) \otimes \mathbb{C}\{L\} \\ &= S(\hat{\mathfrak{h}}^-) \otimes \mathbb{C}[L] \text{ (linearly)} \end{aligned}$$

which is naturally acted upon by \hat{L} , $\hat{\mathfrak{h}}_{\mathbb{Z}}$, \mathfrak{h} , and x^h ($h \in \mathfrak{h}$) as described in previous work ([CalLM4], [L1], [FLM2]).

For $v = h_1(-m_1) \cdots h_r(-m_r) \otimes \iota(a) \in V_L$ with $h_i \in \mathfrak{h}$, $m_i \in \mathbb{N}$, and $a \in \hat{L}$ we set

$$(2.26) \quad Y(v, x) = \circ \frac{1}{(m_1 - 1)!} \left(\frac{d}{dx} \right)^{m_1 - 1} h_1(x) \cdots \frac{1}{(m_r - 1)!} \left(\frac{d}{dx} \right)^{m_r - 1} h_r(x) Y(\iota(a), x) \circ$$

with $h_i(x) = \sum_{m \in \mathbb{Z}} h_i(m) x^{-m-1}$ where we have associated $h_i \otimes t^m = h_i(m)$. We also have

$$(2.27) \quad Y(\iota(a), x) = E^-(\bar{a}, x) E^+(\bar{a}, x) a x^{\bar{a}}$$

where

$$(2.28) \quad E^\pm(\alpha, x) = \exp \left(\sum_{m \in \pm \mathbb{Z}_+} \frac{\alpha(m)}{m} x^{-m} \right) \in (\text{End } V_L)[[x, x^{-1}]]$$

for $\alpha \in \mathfrak{h}$. By Theorem 6.5.3 of [LL], $(V_L, Y, \mathbb{1}, \omega)$ is a simple vertex algebra of central charge $d = \text{rank } L = \dim \mathfrak{h}$ with $\mathbb{1} = 1 \otimes 1$ and

$$(2.29) \quad \omega = \frac{1}{2} \sum_{i=1}^d h_i(-1) h_i(-1) \mathbb{1}$$

where $\{h_1, \dots, h_d\}$ forms an orthonormal basis of \mathfrak{h} . We also extend the automorphism $\hat{\nu}$ of \hat{L} to an automorphism (which we also denote by $\hat{\nu}$) of V_L . Using the tensor product construction of V_L (2.25) we take this extension to be $\nu \otimes \hat{\nu}$. We refer the reader to Section 2 of [CalLM4] ([L1]) for the details showing that this indeed defines an automorphism of V_L .

Our next objective is to construct a $\hat{\nu}$ -twisted V_L -module in the spirit of [CalLM4] and [L1]. We skip many details and focus on how this construction specializes to our case.

Set

$$(2.30) \quad \mathfrak{h}_{(m)} = \{h \in \mathfrak{h} | \hat{\nu} h = (-1)^m h\}$$

for $m \in \mathbb{Z}$, identify $\mathfrak{h}_{(m)} = \mathfrak{h}_{(m \bmod 2)}$, and write

$$(2.31) \quad \mathfrak{h} = \mathfrak{h}_{(0)} \oplus \mathfrak{h}_{(1)}.$$

Observe that

$$(2.32) \quad \mathfrak{h}_{(0)} = \{\alpha + \nu\alpha \mid \alpha \in \mathfrak{h}\}$$

and

$$(2.33) \quad \mathfrak{h}_{(1)} = \{\alpha - \nu\alpha \mid \alpha \in \mathfrak{h}\}.$$

Let P_0 and P_1 be the projections of \mathfrak{h} onto $\mathfrak{h}_{(0)}$ and $\mathfrak{h}_{(1)}$ respectively. For $h \in \mathfrak{h}$ and $m \in \mathbb{Z}/2\mathbb{Z}$ we set $h_{(m)} = P_m h$. Thus, we may write

$$(2.34) \quad h_{(m)} = \frac{1}{2}(h + (-1)^m \nu h).$$

We form the ν -twisted affine Lie algebra associated to \mathfrak{h} :

$$(2.35) \quad \hat{\mathfrak{h}}[\nu] = \mathfrak{h}_{(0)} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathfrak{h}_{(1)} \otimes t^{\frac{1}{2}} \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}k$$

where k is central and

$$(2.36) \quad [\alpha \otimes t^r, \beta \otimes t^s] = \langle \alpha, \beta \rangle r \delta_{r+s,0} k$$

for $r, s \in \frac{1}{2}\mathbb{Z}$, $\alpha \in \mathfrak{h}_{(2r)}$, and $\beta \in \mathfrak{h}_{(2s)}$. We have that $\hat{\mathfrak{h}}[\nu]$ is $\frac{1}{2}\mathbb{Z}$ -graded by weights such that

$$(2.37) \quad \text{wt}(\alpha \otimes t^m) = -m \quad \text{and} \quad \text{wt}(k) = 0.$$

We set

$$(2.38) \quad N = (1 - P_0)\mathfrak{h} \cap L$$

and observe that in our setting we have

$$(2.39) \quad N = \prod_{i=1}^l \mathbb{Z}(\alpha_i - \nu\alpha_i).$$

Using Proposition 6.2 of [L1], let \mathbb{C}_τ denote the one dimensional \hat{N} -module \mathbb{C} with character τ and write

$$(2.40) \quad T = \mathbb{C}_\tau.$$

Now consider the induced \hat{L}_ν -module

$$(2.41) \quad U_T = \mathbb{C}[\hat{L}_\nu] \otimes_{\mathbb{C}[\hat{N}]} T \cong \mathbb{C}[L/N],$$

which is graded by weights and on which \hat{L}_ν , $\mathfrak{h}_{(0)}$, and x^h for $h \in \mathfrak{h}_{(0)}$ all naturally act. Set

$$(2.42) \quad V_L^T = S[\nu] \otimes U_T \cong S(\hat{\mathfrak{h}}[\nu]^-) \otimes \mathbb{C}[L/N],$$

which is naturally acted upon by \hat{L}_ν , $\hat{\mathfrak{h}}_{\frac{1}{2}\mathbb{Z}}$, $\mathfrak{h}_{(0)}$, and x^h for $h \in \mathfrak{h}$.

For each $\alpha \in \mathfrak{h}$ and $m \in \frac{1}{2}\mathbb{Z}$ define the operators on V_L^T

$$(2.43) \quad \alpha_{(2m)} \otimes t^m \mapsto \alpha^{\hat{\nu}}(m)$$

and set

$$(2.44) \quad \alpha^{\hat{\nu}}(x) = \sum_{m \in \frac{1}{2}\mathbb{Z}} \alpha^{\hat{\nu}}(m) x^{-m-1}.$$

Of most importance will be the $\hat{\nu}$ -twisted vertex operators acting on V_L^T for each $e_\alpha \in \hat{L}$

$$(2.45) \quad Y^{\hat{\nu}}(\iota(e_\alpha), x) 2^{-\frac{\langle \alpha, \alpha \rangle}{2}} E^-(-\alpha, x) E^+(-\alpha, x) e_\alpha x^{\alpha(0) + \frac{\langle \alpha(0), \alpha(0) \rangle}{2} - \frac{\langle \alpha, \alpha \rangle}{2}}.$$

as defined in [L1], where

$$(2.46) \quad E^\pm(-\alpha, x) = \exp \left(\sum_{n \in \pm \frac{1}{2}\mathbb{Z}_+} \frac{-\alpha(2n)(n)}{n} x^{-n} \right),$$

for $\alpha \in \mathfrak{h}$. For $m \in \frac{1}{2}$ and $\alpha \in L$ define the component operators $x_\alpha^{\hat{\nu}}(m)$ by

$$(2.47) \quad Y^{\hat{\nu}}(\iota(e_\alpha), x) = \sum_{m \in \frac{1}{2}\mathbb{Z}} x_\alpha^{\hat{\nu}}(m) x^{-m - \frac{\langle \alpha, \alpha \rangle}{2}}.$$

Our construction to this point has been fairly general, apart from the restrictions given by (2.2) and (2.3). From now on we will assume that L is a root lattice of type A_{2n-1} , D_n , or E_6 . In the case that we have a root lattice of type A or D we make the standard choice of simple roots, which we now recall. In the case of A_{2n-1} , we let

$$(2.48) \quad L = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_{2n-1} \text{ with } \alpha_i = \varepsilon_i - \varepsilon_{i-1}$$

where $\{\varepsilon_i\}$ is the standard basis of \mathbb{R}^{2n} . This corresponds to the following standard labeling of the Dynkin diagram.

$$\begin{array}{ccccccc} \circ & - & \circ & - & \cdots & - & \circ & - & \circ & - & \circ & - & \cdots & - & \circ & - & \circ \\ \alpha_1 & & \alpha_2 & & & & \alpha_{n-1} & & \alpha_n & & \alpha_{n+1} & & & & \alpha_{2n-2} & & \alpha_{2n-1} \end{array}$$

In this case, our isometry ν corresponds to the Dynkin diagram folding and is defined on the simple roots by

$$(2.49) \quad \nu\alpha_i = \alpha_{2n-i} \text{ for } 1 \leq i \leq 2n-1,$$

and extended \mathbb{Z} -linearly. Observe that α_n is fixed under ν . Moreover, the corresponding set of roots is given by

$$(2.50) \quad \Delta = \{\varepsilon_i - \varepsilon_j | 1 \leq i \neq j \leq 2n\},$$

where we choose the following subset of positive roots

$$(2.51) \quad \Delta_+ = \{\varepsilon_i - \varepsilon_j | 1 \leq i < j \leq 2n\}$$

so that we have

$$(2.52) \quad \Delta = \Delta_+ \cup (-\Delta_+).$$

For $1 \leq i < j \leq 2n$ we may write $\pm(\varepsilon_i - \varepsilon_j) = \pm(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1})$ from which we have

$$(2.53) \quad \nu(\pm(\varepsilon_i - \varepsilon_j)) = \pm(\varepsilon_{2n-j+1} - \varepsilon_{2n-i+1}).$$

In the case of D_n , we have

$$(2.54) \quad L = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n \text{ with } \alpha_i = \varepsilon_i - \varepsilon_{i+1} \text{ for } 1 \leq i < n \\ \text{and } \alpha_n = \varepsilon_{n-1} + \varepsilon_n,$$

where $\{\varepsilon_i\}$ is the standard basis of \mathbb{R}^n . This corresponds to the following standard labeling of the Dynkin diagram

$$\begin{array}{ccccccc} & & & & \circ & & \\ & & & & | & & \alpha_n \\ \circ & - & \circ & - & \cdots & - & \circ & - & \circ \\ \alpha_1 & & \alpha_2 & & & & \alpha_{n-2} & & \alpha_{n-1} \end{array}$$

In this case, ν is defined by

$$(2.55) \quad \begin{aligned} \nu\alpha_i &= \alpha_i \text{ for } 1 \leq i \leq n-2 \\ \nu\alpha_{n-1} &= \alpha_n \\ \nu\alpha_n &= \alpha_{n-1}, \end{aligned}$$

and extended \mathbb{Z} -linearly. Moreover, the corresponding set of roots is given by

$$(2.56) \quad \Delta = \{\varepsilon_i \pm \varepsilon_j | 1 \leq i \neq j \leq n\},$$

where we choose the following subset of positive roots

$$(2.57) \quad \Delta_+ = \{\varepsilon_i \pm \varepsilon_j | 1 \leq i < j \leq n\}$$

so that we have

$$(2.58) \quad \Delta = \Delta_+ \cup (-\Delta_+).$$

We follow a procedure similar to that above to determine that the action of ν on any root is

$$(2.59) \quad \nu(\pm(\varepsilon_i \pm \varepsilon_j)) = \pm(\varepsilon_i \pm (-1)^{\delta_{j,n}} \varepsilon_j),$$

for $1 \leq i < j \leq n$.

In the case of E_6 , we first consider the standard basis for the root system with simple roots given by

$$(2.60) \quad \begin{aligned} \alpha_1 &= \varepsilon_1 - \varepsilon_2 \\ \alpha_2 &= \varepsilon_2 - \varepsilon_3 \\ \alpha_3 &= \varepsilon_3 - \varepsilon_4 \\ \alpha_4 &= \varepsilon_4 + \varepsilon_5 \\ \alpha_5 &= -\frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5) + \frac{\sqrt{3}}{2}\varepsilon_6 \\ \alpha_6 &= \varepsilon_4 - \varepsilon_5, \end{aligned}$$

where $\{\varepsilon_1, \dots, \varepsilon_6\}$ is the standard basis of \mathbb{R}^6 . We now consider the invertible change of basis matrix

$$(2.61) \quad P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2\sqrt{3}} \\ -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2\sqrt{3}} \\ -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2\sqrt{3}} \\ -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2\sqrt{3}} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & 0 & 0 & 0 \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \end{pmatrix}$$

and observe that for all $\mathbf{v}, \mathbf{w} \in \Delta_{E_6}$, where Δ_{E_6} is the set of roots of E_6 , we have

$$(2.62) \quad \langle P\mathbf{v}, P\mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle,$$

where on the left we take the standard inner product on \mathbb{R}^9 and on the right the standard inner product on \mathbb{R}^6 . The result is that a copy of the E_6 root system may be embedded into \mathbb{R}^9 . Now we make the identification $\mathbb{R}^9 = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ and observe that result is a root system made up of nine dimensional vectors of the form $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$. The positive roots are of two main types, either one of $\mathbf{v}_k = \varepsilon_i - \varepsilon_j$ with $1 \leq i < j \leq 3$ while the other two are the zero vector, or one of the

27 combinations where each \mathbf{v}_i is taken from $\{-\frac{2}{3}\varepsilon_1 + \frac{1}{3}\varepsilon_2 + \frac{1}{3}\varepsilon_3, \frac{1}{3}\varepsilon_1 - \frac{2}{3}\varepsilon_2 + \frac{1}{3}\varepsilon_3, \frac{1}{3}\varepsilon_1 + \frac{1}{3}\varepsilon_2 - \frac{2}{3}\varepsilon_3\}$. With this identification we relabel the simple roots to be

$$\begin{aligned}
 \alpha_1 &= (\mathbf{0}, \mathbf{0}, \varepsilon_2 - \varepsilon_3) \\
 \alpha_2 &= (\mathbf{0}, \mathbf{0}, \varepsilon_1 - \varepsilon_2) \\
 \alpha_3 &= (\frac{1}{3}\varepsilon_1 - \frac{2}{3}\varepsilon_2 + \frac{1}{3}\varepsilon_3, -\frac{2}{3}\varepsilon_1 + \frac{1}{3}\varepsilon_2 + \frac{1}{3}\varepsilon_3, -\frac{2}{3}\varepsilon_1 + \frac{1}{3}\varepsilon_2 + \frac{1}{3}\varepsilon_3) \\
 \alpha_4 &= (\mathbf{0}, \varepsilon_1 - \varepsilon_2, \mathbf{0}) \\
 \alpha_5 &= (\mathbf{0}, \varepsilon_2 - \varepsilon_3, \mathbf{0}) \\
 \alpha_6 &= (\varepsilon_2 - \varepsilon_3, \mathbf{0}, \mathbf{0}).
 \end{aligned}
 \tag{2.63}$$

This corresponds to the following labeling of the Dynkin diagram.

$$\begin{array}{ccccccc}
 & & \circ & & & & \\
 & & | & & & & \\
 \circ & - & \circ & - & \circ & - & \circ & - & \circ \\
 \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 & & \alpha_5
 \end{array}$$

Here we have

$$\begin{aligned}
 \nu\alpha_1 &= \alpha_5 \\
 \nu\alpha_2 &= \alpha_4 \\
 \nu\alpha_3 &= \alpha_3 \\
 \nu\alpha_6 &= \alpha_6.
 \end{aligned}
 \tag{2.64}$$

Using this choice for the root vectors the action of ν is given by

$$\nu(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = (\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_2).
 \tag{2.65}$$

In each of these cases, we take $\langle \cdot, \cdot \rangle : L \times L \rightarrow \mathbb{Z}$ to be the bilinear form given by

$$\langle \alpha, \beta \rangle = 2 \frac{(\beta, \alpha)}{(\beta, \beta)},
 \tag{2.66}$$

where (\cdot, \cdot) is the standard Euclidean inner product given by $(\varepsilon_i, \varepsilon_j) = \delta_{i,j}$. Recall from (2.17), in each of these cases the isometry ν lifts to an automorphism $\hat{\nu}$ of V_L where

$$\hat{\nu}e_{\alpha_i} = e_{\nu\alpha_i}.
 \tag{2.67}$$

Now we focus on the construction of the Lie algebras of type A_{2n-1} , D_n , and E_6 , as well as their affine twisted counterparts from the appropriate root lattice and isometry ν . Each of these cases will be handled simultaneously and we step back to point out significant differences when needed. Consider the vector space defined by

$$\mathfrak{g} = \mathfrak{h} \oplus \coprod_{\alpha \in \Delta} x_\alpha,
 \tag{2.68}$$

where \mathfrak{h} is defined as in (2.18) and $\{x_\alpha\}$ is a set of symbols, where Δ is the set of roots corresponding to L .

We endow \mathfrak{g} with the structure of a Lie algebra via the bracket defined

$$[h, x_\alpha] = \langle h, \alpha \rangle x_\alpha, \quad [\mathfrak{h}, \mathfrak{h}] = 0
 \tag{2.69}$$

where $h \in \mathfrak{h}$ and $\alpha \in \Delta$ and

$$[x_\alpha, x_\beta] = \begin{cases} \epsilon_{C_0}(\alpha, -\alpha)\alpha & \text{if } \alpha + \beta = 0 \\ \epsilon_{C_0}(\alpha, \beta)x_{\alpha+\beta} & \text{if } \alpha + \beta \in \Delta \\ 0 & \text{otherwise} \end{cases}
 \tag{2.70}$$

Observe that \mathfrak{g} is a Lie algebra isomorphic to one of type A_{2n-1} , D_n , or E_6 depending on the choice of L . We extend the bilinear form $\langle \cdot, \cdot \rangle$ to \mathfrak{g} by

$$(2.71) \quad \langle h, x_\alpha \rangle = \langle x_\alpha, h \rangle = 0$$

and

$$(2.72) \quad \langle x_\alpha, x_\beta \rangle = \begin{cases} \epsilon_{C_0}(\alpha, -\alpha) & \text{if } \alpha + \beta = 0 \\ 0 & \text{if } \alpha + \beta \neq 0 \end{cases}$$

Following [L1] and [CalLM4], we use our extension of $\nu : L \rightarrow L$ to $\hat{\nu} : \hat{L} \rightarrow \hat{L}$ to lift the automorphism $\nu : \mathfrak{h} \rightarrow \mathfrak{h}$ to an automorphism $\hat{\nu} : \mathfrak{g} \rightarrow \mathfrak{g}$ by setting

$$(2.73) \quad \hat{\nu}x_\alpha = \psi(\alpha)x_{\nu\alpha}$$

for all $\alpha \in \Delta$, where ψ is defined in (2.15). Here, we are using our particular choices of $\hat{\nu}$ (extended to $\mathbb{C}\{L\}$) and our section e .

For $m \in \mathbb{Z}$ set

$$(2.74) \quad \mathfrak{g}_{(m)} = \{x \in \mathfrak{g} | \hat{\nu}(x) = (-1)^m x\}.$$

Now we decompose \mathfrak{g} into eigenspaces of $\hat{\nu}$ as follows

$$(2.75) \quad \mathfrak{g}_{(0)} = \coprod_{i=1}^l \mathbb{C}(\alpha_i + \nu\alpha_i) \oplus \coprod_{\alpha \in \Delta} \mathbb{C}(x_\alpha + x_{\nu\alpha})$$

and

$$(2.76) \quad \mathfrak{g}_{(1)} = \coprod_{i=1}^l \mathbb{C}(\alpha_i - \nu\alpha_i) \oplus \coprod_{\alpha \in \Delta} \mathbb{C}(x_\alpha - x_{\nu\alpha})$$

Form the $\hat{\nu}$ -twisted affine Lie algebra associated to \mathfrak{g} and $\hat{\nu}$:

$$(2.77) \quad \hat{\mathfrak{g}}[\hat{\nu}] = \coprod_{n \in \mathbb{Z}} \mathfrak{g}_{(n)} \otimes t^{n/2} \oplus \mathbb{C}k = \coprod_{n \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_{(2n)} \otimes t^n \oplus \mathbb{C}k$$

with

$$(2.78) \quad [x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + \langle x, y \rangle m \delta_{m+n, 0} k$$

and

$$(2.79) \quad [k, \hat{\mathfrak{g}}[\hat{\nu}]] = 0,$$

for $m, n \in \frac{1}{2}\mathbb{Z}$, $x \in \mathfrak{g}_{(2m)}$, and $y \in \mathfrak{g}_{(2n)}$. Set

$$(2.80) \quad \tilde{\mathfrak{g}}[\hat{\nu}] = \hat{\mathfrak{g}}[\hat{\nu}] \oplus \mathbb{C}d,$$

with

$$(2.81) \quad [d, x \otimes t^n] = nx \otimes t^n,$$

for $x \in \mathfrak{g}_{(2n)}$, $n \in \frac{1}{2}\mathbb{Z}$ and $[d, k] = 0$. The Lie algebra $\tilde{\mathfrak{g}}[\hat{\nu}]$ is isomorphic to $A_{2n-1}^{(2)}$, $D_n^{(2)}$, or $E_6^{(2)}$ depending on the choice of L , and is $\frac{1}{2}\mathbb{Z}$ -graded.

We now recall the following result that gives V_L^T the structure of a $\hat{\mathfrak{g}}[\hat{\nu}]$ -module:

Theorem 2.1. (*Theorem 3.1 [CalLM4], Theorem 9.1 [L1], Theorem 3 [FLM2]*) *The representation of $\hat{\mathfrak{h}}[\nu]$ on V_L^T extends uniquely to a Lie algebra representation of $\hat{\mathfrak{g}}[\hat{\nu}]$ on V_L^T such that*

$$(x_\alpha)_{(2n)} \otimes t^n \mapsto x_\alpha^{\hat{\nu}}(n)$$

for all $m \in \frac{1}{2}\mathbb{Z}$ and $\alpha \in L$. Moreover V_L^T is irreducible as a $\hat{\mathfrak{g}}[\hat{\nu}]$ -module.

As in Section 2 of [CalLM4] (also Section 6 of [L1]) we have a tensor product grading on V_L^T given by the action of $L^{\hat{\nu}}(0)$ with

$$\begin{aligned} L^{\hat{\nu}}(0)1 &= \frac{1}{16} \sum_{j=1}^1 j(2-j) \dim \mathfrak{h}_{(j)} 1 \\ (2.82) \quad &= \frac{1}{16} (\dim \mathfrak{h}_{(1)}), \end{aligned}$$

Where

$$(2.83) \quad Y^{\hat{\nu}}(\omega, x) = \sum_{m \in \mathbb{Z}} L^{\nu}(m) x^{-n-2}.$$

We will write $\text{wt}(1) = \frac{\dim \mathfrak{h}_{(1)}}{16}$. For a homogeneous element $v \in V_L^T$ we have

$$(2.84) \quad \hat{L}^{\hat{\nu}}(0)v = \left(\text{wt}(v) + \frac{\dim \mathfrak{h}_{(1)}}{16} \right) v.$$

For each of our special cases we have

$$\begin{aligned} (2.85) \quad \text{wt}(1) &= \frac{n-1}{16}, \text{ for } A_{2n-1} \\ \text{wt}(1) &= \frac{1}{16}, \text{ for } D_n \\ \text{wt}(1) &= \frac{1}{8}, \text{ for } E_6, \end{aligned}$$

which carries over to the weight of any homogeneous element in (2.84).

By Proposition 6.3 of [DL1] we have that

$$\begin{aligned} (2.86) \quad [Y^{\hat{\nu}}(\omega, x_1), Y^{\hat{\nu}}(\iota(e_\alpha), x_2)] \\ = x_2^{-1} \frac{d}{dx_2} Y^{\hat{\nu}}(\iota(e_\alpha), x_2) \delta(x_1/x_2) - \frac{1}{2} \langle \alpha, \alpha \rangle x_2^{-1} Y^{\hat{\nu}}(\iota(e_\alpha), x_2) \frac{d}{dx_1} \delta(x_1/x_2), \end{aligned}$$

where $\delta(x) = \sum_{n \in \mathbb{Z}} x^n$. Taking $\text{Res}_{x_2} \text{Res}_{x_1}$ of $x_1 x_2^m$ times of both sides of (2.86) immediately gives

$$(2.87) \quad [L^{\hat{\nu}}(0), x_\alpha^{\hat{\nu}}(m)] = \left(-m - 1 + \frac{1}{2} \langle \alpha, \alpha \rangle \right) x_\alpha^{\hat{\nu}}(m)$$

and thus

$$(2.88) \quad \text{wt}(x_\alpha^{\hat{\nu}}(m)) = -m - 1 + \frac{1}{2} \langle \alpha, \alpha \rangle.$$

Let

$$(2.89) \quad \mathcal{B}^* = \{\lambda_i \in \mathbb{Q} \otimes_{\mathbb{Z}} L \mid 1 \leq i \leq l \text{ and } \langle \lambda_i, \alpha_j \rangle = \delta_{i,j}\}$$

be the basis of the lattice dual to L with respect to the basis $\mathcal{B} = \{\alpha_i \mid 1 \leq i \leq l\}$. The λ_i are in the rational span of \mathcal{B} and so using (2.32) we have

$$(2.90) \quad (\lambda_i)_{(0)} = \frac{1}{2}(\lambda_i + \nu \lambda_i).$$

We now use the eigenvalues of the operators $(\lambda_i)_{(0)}(0)$ to construct a new charge grading as follows. Let

$$(2.91) \quad \mathfrak{o}_i = \{\lambda \in \mathcal{B}^* \mid \lambda = \nu^m \lambda_i \text{ for some } m \in \mathbb{N}\},$$

that is, \mathfrak{o}_i is the orbit of λ_i under the action of ν . Observe that the distinct \mathfrak{o}_i form a partition of \mathcal{B}^* . Now define

$$(2.92) \quad \lambda^{(i)} = \sum_{\lambda \in \mathfrak{o}_i} (\lambda)_{(0)}.$$

Finally, we define our charge grading to be a tuple built from the eigenvalues of $\lambda^{(i)}(0)$ as i runs over all distinct orbits \mathfrak{o}_i . That is

$$(2.93) \quad \text{ch}(x_\alpha^{\hat{\nu}}(m)) = \left(\langle \alpha, \lambda^{(i_1)} \rangle, \langle \alpha, \lambda^{(i_2)} \rangle, \dots, \langle \alpha, \lambda^{(i_k)} \rangle \right),$$

where the sets \mathfrak{o}_{i_j} are all distinct (and disjoint), with $i_1 < i_2 < \dots < i_k$. It is easy to see that

$$(2.94) \quad \text{ch}(x_\alpha^{\hat{\nu}}(m)) = (d_{i_1} \langle \alpha, (\lambda_{i_1})_{(0)} \rangle, d_{i_2} \langle \alpha, (\lambda_{i_2})_{(0)} \rangle, \dots, d_{i_k} \langle \alpha, (\lambda_{i_k})_{(0)} \rangle),$$

where $d_{i_j} = |\mathfrak{o}_{i_j}|$. Further if $\text{ch}(v) = (m_1, \dots, m_k)$ then we say the total charge of v is $m_1 + m_2 + \dots + m_k$.

In this case of A_{2n-1} the amounts to the n -tuple given by

$$(2.95) \quad \text{ch}(x_\alpha^{\hat{\nu}}(m)) = \langle 2 \langle \alpha, (\lambda_1)_{(0)} \rangle, \dots, 2 \langle \alpha, (\lambda_{n-1})_{(0)} \rangle, \langle \alpha, (\lambda_n)_{(0)} \rangle \rangle.$$

In the case of D_n we have

$$(2.96) \quad \text{ch}(x_\alpha^{\hat{\nu}}(m)) = \langle \langle \alpha, (\lambda_1)_{(0)} \rangle, \dots, \langle \alpha, (\lambda_{n-2})_{(0)} \rangle, 2 \langle \alpha, (\lambda_{n-1})_{(0)} \rangle \rangle.$$

Finally in the E_6 case we have

$$(2.97) \quad \text{ch}(x_\alpha^{\hat{\nu}}(m)) = \langle 2 \langle \alpha, (\lambda_1)_{(0)} \rangle, 2 \langle \alpha, (\lambda_2)_{(0)} \rangle, \langle \alpha, (\lambda_3)_{(0)} \rangle, \langle \alpha, (\lambda_6)_{(0)} \rangle \rangle.$$

Consider the following ν -stable Lie subalgebra of \mathfrak{g} :

$$(2.98) \quad \mathfrak{n} = \coprod_{\alpha \in \Delta_+} \mathbb{C} x_\alpha.$$

and its $\hat{\nu}$ -twisted affinization

$$(2.99) \quad \hat{\mathfrak{n}}[\hat{\nu}] = \coprod_{m \in \mathbb{Z}} \mathfrak{n}_{(m)} \otimes t^{m/2} \oplus \mathbb{C} k = \coprod_{m \in \frac{1}{2}\mathbb{Z}} \mathfrak{n}_{(2m)} \otimes t^m \oplus \mathbb{C} k,$$

where $\mathfrak{n}_{(m)}$ is the $(-1)^m$ -eigenspace of $\hat{\nu}$ in \mathfrak{n} . As in [CalLM4], [CalMPe], and [PS], we take

$$(2.100) \quad \begin{aligned} \bar{\mathfrak{n}}[\hat{\nu}] &= \coprod_{m \in \frac{1}{2}\mathbb{Z}} \mathfrak{n}_{(2m)} \otimes t^m \\ \bar{\mathfrak{n}}[\hat{\nu}]_+ &= \coprod_{m \in \frac{1}{2}\mathbb{Z}_{\geq 0}} \mathfrak{n}_{(2m)} \otimes t^m \\ \bar{\mathfrak{n}}[\hat{\nu}]_- &= \coprod_{m \in \frac{1}{2}\mathbb{Z}_{< 0}} \mathfrak{n}_{(2m)} \otimes t^m \end{aligned}$$

Now we will investigate the properties of operators from $\hat{\mathfrak{n}}[\hat{\nu}]$ acting on V_L^T by investigating the vertex operators (2.47) and their modes. Using equation (2.93) of [CalLM4], we have that

$$(2.101) \quad Y^{\hat{\nu}}(\hat{\nu}v, x) = \lim_{x^{\frac{1}{2}} \rightarrow (-1)x^{\frac{1}{2}}} Y^{\hat{\nu}}(v, x).$$

In our setting (ADE), we have two cases to consider: when α is fixed under ν or when α is not fixed under ν , with $\nu^2\alpha = \alpha$. In the case that α is fixed we have

$$(2.102) \quad Y^{\hat{\nu}}(\iota(e_\alpha), x) = \lim_{x^{\frac{1}{2}} \rightarrow (-1)x^{\frac{1}{2}}} Y^{\hat{\nu}}(\iota(e_\alpha), x),$$

and thus

$$(2.103) \quad Y^{\hat{\nu}}(\iota(e_\alpha), x) \in (\text{End } V_L^T)[[x, x^{-1}]].$$

In terms of component operators this implies that

$$(2.104) \quad x_{\nu_\alpha}^{\hat{\nu}}(m) = 0 \text{ for all } m \in \frac{1}{2} + \mathbb{Z}.$$

In the other case we have that

$$(2.105) \quad Y^{\hat{\nu}}(\iota(e_{\nu\alpha}), x) = \lim_{x^{\frac{1}{2}} \rightarrow (-1)x^{\frac{1}{2}}} Y^{\hat{\nu}}(\iota(e_\alpha), x),$$

and thus

$$(2.106) \quad \begin{aligned} x_{\nu_\alpha}^{\hat{\nu}}(m) &= x_{\nu_\alpha}^{\hat{\nu}}(m) \text{ for } m \in \mathbb{Z} \\ x_{\nu_\alpha}^{\hat{\nu}}(m) &= -x_{\nu_\alpha}^{\hat{\nu}}(m) \text{ for } m \in \frac{1}{2} + \mathbb{Z}. \end{aligned}$$

In our setting, the commutator formula among twisted vertex operators becomes

$$(2.107) \quad [Y^{\hat{\nu}}(u, x_1), Y^{\hat{\nu}}(v, x_2)] = \frac{1}{2} x_2^{-1} \text{Res}_{x_0} \left(\sum_{j \in \mathbb{Z}/2\mathbb{Z}} \delta \left((-1)^j \frac{(x_1 - x_0)^{1/2}}{x_2^{1/2}} \right) Y^{\hat{\nu}}(Y(\hat{\nu}^j u, x_0)v, x_2) \right).$$

As a consequence, using the fact that $\langle \alpha, \beta \rangle \geq -1$ for all $\alpha, \beta \in \Delta_+$ we have the following commutator formula for twisted vertex operators

$$(2.108) \quad [Y^{\hat{\nu}}(\iota(e_\alpha), x_1), Y^{\hat{\nu}}(\iota(e_\beta), x_2)] = x_2^{-1} \frac{1}{2} \sum_{j \in \mathbb{Z}/2\mathbb{Z}} \delta \left((-1)^j \frac{x_1^{\frac{1}{2}}}{x_2^{\frac{1}{2}}} \right) Y^{\hat{\nu}}(x_{\nu^j \alpha}(0) \iota(e_\beta), x_2),$$

leading to the following lemma:

Lemma 2.1. *For $\langle \alpha, \beta \rangle \in \Delta^+$ we have*

$$(2.109) \quad [Y^{\hat{\nu}}(\iota(e_\alpha), x_1), Y^{\hat{\nu}}(\iota(e_\beta), x_2)] = 0 \text{ when } \langle \nu^j \alpha, \beta \rangle \geq 0 \text{ for } 0 \leq j \leq 1$$

$$(2.110) \quad \begin{aligned} [Y^{\hat{\nu}}(\iota(e_\alpha), x_1), Y^{\hat{\nu}}(\iota(e_\beta), x_2)] &= x_2^{-1} \frac{1}{2} \epsilon_{C_0}(\alpha, \beta) \delta \left(\frac{x_1^{\frac{1}{2}}}{x_2^{\frac{1}{2}}} \right) Y^{\hat{\nu}}(\iota(e_{\alpha+\beta}), x_2) \\ &\text{when } \langle \alpha, \beta \rangle = -1 \text{ and } \langle \nu \alpha, \beta \rangle \geq 0 \end{aligned}$$

$$(2.111) \quad \begin{aligned} [Y^{\hat{\nu}}(\iota(e_\alpha), x_1), Y^{\hat{\nu}}(\iota(e_\beta), x_2)] &= x_2^{-1} \frac{1}{2} \sum_{j=0}^1 \epsilon_{C_0}(\nu^j \alpha, \beta) \delta \left((-1)^j \frac{x_1^{\frac{1}{2}}}{x_2^{\frac{1}{2}}} \right) Y^{\hat{\nu}}(\iota(e_{\nu^j \alpha + \beta}), x_2) \\ &\text{when } \langle \alpha, \beta \rangle = \langle \nu \alpha, \beta \rangle = -1. \end{aligned}$$

Proof. This follows from the standard vertex algebra fact that $x_\alpha (-\langle \alpha, \beta \rangle - 1) e_\beta = \epsilon_{C_0}(\alpha, \beta) e_{\alpha+\beta}$. \square

We can use Lemma 2.1 to make the following statement regarding component operators.

Corollary 2.1. *For $\alpha, \beta \in \Delta_+$ and $r, s \in \frac{1}{2}\mathbb{Z}$ chosen from a suitable subset so that the operators are nonzero (see (2.104)), we have*

$$(2.112) \quad [x_\alpha^{\hat{\nu}}(r), x_\beta^{\hat{\nu}}(s)] = 0 \text{ when } \langle \alpha, \beta \rangle \geq 0 \text{ and } \langle \nu \alpha, \beta \rangle \geq 0,$$

$$(2.113) \quad [x_\alpha^{\hat{\nu}}(r), x_\beta^{\hat{\nu}}(s)] = \frac{1}{2} \epsilon_{C_0}(\alpha, \beta) x_{\alpha+\beta}^{\hat{\nu}}(r+s) \text{ when } \langle \alpha, \beta \rangle = -1 \text{ and } \langle \nu \alpha, \beta \rangle \geq 0,$$

and

(2.114)

$$[x_{\alpha}^{\dot{\nu}}(r), x_{\beta}^{\dot{\nu}}(s)] = \frac{1}{2}\epsilon_{C_0}(\alpha, \beta)x_{\alpha+\beta}^{\dot{\nu}}(r+s) + (-1)^{2r}\frac{1}{2}\epsilon_{C_0}(\nu\alpha, \beta)x_{\nu\alpha+\beta}^{\dot{\nu}}(r+s) \text{ when } \langle\alpha, \beta\rangle = -1 \text{ and } \langle\nu\alpha, \beta\rangle = -1.$$

We now present two technical Lemmas that will be of use in the simplification of (2.114).

Lemma 2.2. *Suppose that $\alpha, \beta \in \Delta_+$ such that $\langle\alpha, \beta\rangle = \langle\nu\alpha, \beta\rangle = -1$. Then either $\nu\alpha = \alpha$ or $\nu\beta = \beta$.*

Proof. We separate our argument into cases depending on type. First considering the A_{2n-1} case.

Suppose $\alpha, \beta \in \Delta_+$ are such that $\langle\alpha, \beta\rangle = \langle\nu\alpha, \beta\rangle = -1$. Since we are dealing with positive roots of A_{2n-1} we may write

$$(2.115) \quad \alpha = \varepsilon_i - \varepsilon_j \text{ with } 1 \leq i < j \leq 2n$$

and

$$(2.116) \quad \beta = \varepsilon_k - \varepsilon_l \text{ with } 1 \leq k < l \leq 2n.$$

The isometry ν acts as follows on the roots written in this form

$$(2.117) \quad \nu\alpha = \varepsilon_{2n+1-j} - \varepsilon_{2n+1-i}.$$

Observe that we have

$$(2.118) \quad -1 = \langle\alpha, \beta\rangle = \delta_{i,k} + \delta_{j,l} - \delta_{j,k} - \delta_{i,l}.$$

We claim that only one of $\delta_{j,k}, \delta_{i,l}$ is nonzero. Indeed, otherwise we have

$$(2.119) \quad \beta = \varepsilon_k - \varepsilon_l = \varepsilon_j - \varepsilon_i = \alpha \notin \Delta_+.$$

Thus we have $i \neq k, j \neq l$, and either $j = k$ or $i = l$ (but not both). Without loss of generality we take $j = k$, and thus $i \neq l$. Now considering an expansion of $\langle\nu\alpha, \beta\rangle$ similar to (2.118) we have

$$(2.120) \quad 2n+1-i = j$$

or

$$(2.121) \quad 2n+1-j = l.$$

If (2.120) holds we have $\alpha = \varepsilon_i - \varepsilon_{2n+1-i}$ which is fixed under ν , while if (2.121) holds we have $\beta = \varepsilon_j - \varepsilon_{2n+1-j}$ with is also fixed under ν .

Now we consider the D_n case, here we may write positive roots in the form

$$(2.122) \quad \alpha = \varepsilon_i + (-1)^a \varepsilon_j \text{ with } 1 \leq i < j \leq n$$

and

$$(2.123) \quad \beta = \varepsilon_k + (-1)^b \varepsilon_l \text{ with } 1 \leq k < l \leq n,$$

where $a, b \in \{0, 1\}$. In this setting we have

$$(2.124) \quad \nu\alpha = \varepsilon_i + (-1)^{a+\delta_{j,n}} \varepsilon_j.$$

We may also write

$$(2.125) \quad -1 = \langle\alpha, \beta\rangle = \delta_{i,k} + (-1)^a \delta_{j,k} + (-1)^b \delta_{i,l} + (-1)^{a+b} \delta_{j,l}$$

and

$$(2.126) \quad -1 = \langle\nu\alpha, \beta\rangle = \delta_{i,k} + (-1)^{a+\delta_{j,n}} \delta_{j,k} + (-1)^b \delta_{i,l} + (-1)^{a+b+\delta_{j,n}} \delta_{j,l}.$$

Using this to expand $\langle\alpha, \beta\rangle - \langle\nu\alpha, \beta\rangle = 0$ we have

$$(2.127) \quad ((-1)^a - (-1)^{a+\delta_{j,n}}) \delta_{j,k} - ((-1)^{a+b} - (-1)^{a+b+\delta_{j,n}}) \delta_{j,l} = 0.$$

We now claim that we cannot have both $j = k$ and $j = l$, otherwise we would have $\beta = 2\epsilon_j \notin \Delta_+$. This leaves us with the following

Case 1: $j \neq k$ and $j \neq l$ and thus at least one of j or l is not n and the corresponding root is fixed under ν .

Case 2: $j = k$ and $j \neq l$ and thus $(-1)^a = (-1)^{a+\delta_{j,n}}$ which implies $j \neq n$ and α is fixed.

Case 3: $j \neq k$ and $j = l$ and thus $(-1)^{a+b} = (-1)^{a+b+\delta_{j,n}}$ which implies $j \neq n$ and α is fixed.

Now we consider the E_6 case. Here we use a symmetric version of the root system as described above. We take the root system to be made up of nine dimensional vectors which we consider of the form $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$. The positive roots are of two main types, either one of $\mathbf{v}_i = \epsilon_i - \epsilon_j$ with $1 \leq i < j \leq 3$ while the other two are the zero vector, or one of the 27 combinations where each \mathbf{v}_i is taken from $\{-\frac{2}{3}\epsilon_1 + \frac{1}{3}\epsilon_2 + \frac{1}{3}\epsilon_3, \frac{1}{3}\epsilon_1 - \frac{2}{3}\epsilon_2 + \frac{1}{3}\epsilon_3, \frac{1}{3}\epsilon_1 + \frac{1}{3}\epsilon_2 - \frac{2}{3}\epsilon_3\}$. In this setting we take the roots to be

$$\begin{aligned}
 \alpha_1 &= (\mathbf{0}, \mathbf{0}, \epsilon_2 - \epsilon_3) \\
 \alpha_2 &= (\mathbf{0}, \mathbf{0}, \epsilon_1 - \epsilon_2) \\
 \alpha_3 &= (-\frac{1}{3}\epsilon_1 + \frac{2}{3}\epsilon_2 - \frac{1}{3}\epsilon_3, -\frac{2}{3}\epsilon_1 + \frac{1}{3}\epsilon_2 + \frac{1}{3}\epsilon_3, -\frac{2}{3}\epsilon_1 + \frac{1}{3}\epsilon_2 + \frac{1}{3}\epsilon_3) \\
 \alpha_4 &= (\mathbf{0}, \epsilon_1 - \epsilon_2, \mathbf{0}) \\
 \alpha_5 &= (\mathbf{0}, \epsilon_2 - \epsilon_3, \mathbf{0}) \\
 \alpha_6 &= (\epsilon_2 - \epsilon_3, \mathbf{0}, \mathbf{0}).
 \end{aligned}
 \tag{2.128}$$

We have that

$$\nu(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = (\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_2).
 \tag{2.129}$$

Finally, we let $\alpha = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and $\beta = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ and expand the inner product

$$\langle \alpha, \beta \rangle = \langle \mathbf{v}_1, \mathbf{w}_1 \rangle + \langle \mathbf{v}_2, \mathbf{w}_2 \rangle + \langle \mathbf{v}_3, \mathbf{w}_3 \rangle
 \tag{2.130}$$

and notice that the equation $\langle \alpha, \beta \rangle - \langle \nu\alpha, \beta \rangle = 0$ implies that

$$\langle \mathbf{v}_2 - \mathbf{v}_3, \mathbf{w}_2 - \mathbf{w}_3 \rangle = 0.
 \tag{2.131}$$

Now we make the observation that

$$\mathbf{v}_2 - \mathbf{v}_3, \mathbf{w}_2 - \mathbf{w}_3 \in \{\epsilon_i - \epsilon_j | 1 \leq i, j \leq 3\}.
 \tag{2.132}$$

It is clear that all choices of \mathbf{v}_i and \mathbf{w}_j that make $\alpha, \beta \in \Delta_+$, none of them allow for (2.131) to hold unless $\mathbf{v}_2 - \mathbf{v}_3 = 0$ (and thus α is fixed) or $\mathbf{w}_2 - \mathbf{w}_3 = 0$ (and thus β is fixed). \square

In view of Lemma 2.2, if $\langle \alpha, \beta \rangle = -1$ and $\langle \nu\alpha, \beta \rangle = -1$, then we can write (2.114) as

$$[x_\alpha^\nu(r), x_\beta^\nu(s)] = \left(\frac{1}{2}\epsilon_{C_0}(\alpha, \beta) + (-1)^{2r}\frac{1}{2}\epsilon_{C_0}(\alpha, \beta) \right) x_{\alpha+\beta}^\nu(r+s) = \epsilon_{C_0}(\alpha, \beta) x_{\alpha+\beta}^\nu(r+s)
 \tag{2.133}$$

if $\nu\alpha = \alpha$ (which also implies $r \in \mathbb{Z}$). Finally, if $\nu\alpha \neq \alpha$ (in which case $\nu\beta = \beta$ and $s \in \mathbb{Z}$) we have

$$\begin{aligned}
 [x_\alpha^\nu(r), x_\beta^\nu(s)] &= -[x_\beta^\nu(s), x_\alpha^\nu(r)] \\
 &= -\epsilon_{C_0}(\beta, \alpha) x_{\alpha+\beta}^\nu(r+s).
 \end{aligned}$$

3. PRINCIPAL SUBSPACES

Following [CalLM4] and [CalMPe] (cf. [CalLM1]-[CalLM3], [FS1] - [FS2], and other works involving principal subspaces) we define the principal subspace of a highest weight $\tilde{\mathfrak{g}}[\nu]$ -module:

Definition 3.1. For any standard $\tilde{\mathfrak{g}}[\nu]$ -module V with highest weight vector v , define its principal subspace W to be

$$W = U(\bar{\mathfrak{n}}[\nu]) \cdot v.$$

Denote by W_L^T the principal subspace of the standard $\tilde{\mathfrak{g}}[\nu]$ -module V_L^T ,

$$(3.1) \quad W_L^T = U(\bar{\mathfrak{n}}[\nu]) \cdot v_\Lambda,$$

where $\Lambda \in (\mathfrak{h}_{(0)} \oplus \mathbb{C}k \oplus \mathbb{C}d)^*$ is the fundamental weight of $\tilde{\mathfrak{g}}[\nu]$ defined by $\langle \Lambda, k \rangle = 1$, $\langle \Lambda, \mathfrak{h}_{(0)} \rangle = 0$, and $\langle \Lambda, d \rangle = 0$, and v_Λ is a highest weight vector of V_L^T . We have

$$(3.2) \quad W_L^T = U(\bar{\mathfrak{n}}[\nu]_-) \cdot v_\Lambda.$$

We take $v_\Lambda = 1 = 1 \otimes 1 \in V_L^T (\cong S[\nu] \otimes U_T)$.

Consider the surjection

$$(3.3) \quad \begin{aligned} F_\Lambda : U(\hat{\mathfrak{g}}[\nu]) &\rightarrow V_L^T \\ a &\mapsto a \cdot v_\Lambda \end{aligned}$$

and its restriction

$$(3.4) \quad \begin{aligned} f_\Lambda : U(\bar{\mathfrak{n}}[\nu]) &\rightarrow W_L^T \\ a &\mapsto a \cdot v_\Lambda \end{aligned}$$

As in previous work ([CalLM1]-[CalLM4], [CalMPe], and others), our main goal is to describe the kernel of f_Λ , giving us a presentation of W_L^T .

Following [CalLM4] (cf. [CalLM1]-[CalLM3]) we set

$$(3.5) \quad N_L^T = U(\hat{\mathfrak{g}}[\nu]) \otimes_{U(\hat{\mathfrak{g}}[\nu])_{\geq 0}} \mathbb{C}v_\Lambda^N,$$

i.e. the generalized Verma module where,

$$(3.6) \quad \hat{\mathfrak{g}}[\nu]_{\geq 0} = \coprod_{n \geq 0} \mathfrak{g}_{(n)} \otimes t^{n/2} \oplus \mathbb{C}k.$$

We also use the principal subspaces of the generalized Verma module

$$(3.7) \quad W_L^{T,N} = U(\bar{\mathfrak{n}}[\nu]) \cdot v_\Lambda^N \subset N_L^T.$$

Now we define the following related surjections

$$(3.8) \quad \begin{aligned} F_\Lambda^N : U(\hat{\mathfrak{g}}[\nu]) &\rightarrow N_L^T \\ a &\mapsto a \cdot v_\Lambda^N \\ f_\Lambda^N : U(\bar{\mathfrak{n}}[\nu]) &\rightarrow W_L^{T,N} \\ a &\mapsto a \cdot v_\Lambda^N \end{aligned}$$

as well as

$$(3.9) \quad \begin{aligned} \Pi : N_L^T &\rightarrow V_L^T \\ a \cdot v_\Lambda^N &\mapsto a \cdot v_\Lambda \\ \pi : W_L^{T,N} &\rightarrow W_L^T \\ a \cdot v_\Lambda^N &\mapsto a \cdot v_\Lambda. \end{aligned}$$

We now look at some results involving the vertex operators $Y^{\hat{\nu}}(e^{\alpha}, x)$. These will be the source of our description of $\text{Ker}(f_{\Lambda})$. Using the fact

$$(3.10) \quad E^+(\alpha, x_1)E^-(\beta, x_2) = \prod_{p \in \mathbb{Z}/2\mathbb{Z}} \left(1 - (-1)^p \frac{x_2^{\frac{1}{2}}}{x_1^{\frac{1}{2}}} \right)^{\langle \nu^p \alpha, \beta \rangle} E^-(\beta, x_2)E^+(\alpha, x_1),$$

(cf. [L1], [CalLM4]) we have the following results:

Proposition 3.1. *For $\alpha, \beta \in \Delta^+$ we have*

$$(3.11) \quad Y^{\hat{\nu}}(\iota(e_{\alpha}), x)Y^{\hat{\nu}}(\iota(e_{\beta}), x) = 0, \text{ when } \langle \nu^j \alpha, \beta \rangle \geq 0 \text{ for } 0 \leq j \leq 2$$

$$(3.12) \quad \lim_{x_2^{\frac{1}{2}} \rightarrow x_1^{\frac{1}{2}}} \left(x_1^{\frac{1}{2}} - x_2^{\frac{1}{2}} \right) Y^{\hat{\nu}}(\iota(e_{\alpha}), x_1)Y^{\hat{\nu}}(\iota(e_{\beta}), x_2) = 0, \text{ when } \langle \alpha, \beta \rangle = -1 \text{ and } \langle \nu \alpha, \beta \rangle \geq 0$$

$$(3.13) \quad \lim_{x_2^{\frac{1}{2}} \rightarrow x_1^{\frac{1}{2}}} \left(x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}} \right) Y^{\hat{\nu}}(\iota(e_{\alpha}), x_1)Y^{\hat{\nu}}(\iota(e_{\beta}), x_2) = 0, \text{ when } \langle \alpha, \beta \rangle \geq 0 \text{ and } \langle \nu \alpha, \beta \rangle = -1$$

$$(3.14) \quad \lim_{x_2^{\frac{1}{2}} \rightarrow x_1^{\frac{1}{2}}} (x_1 - x_2) Y^{\hat{\nu}}(\iota(e_{\alpha}), x_1)Y^{\hat{\nu}}(\iota(e_{\beta}), x_2) = 0, \text{ when } \langle \nu^j \alpha, \beta \rangle = -1 \text{ for } 0 \leq j \leq 1.$$

By taking appropriate coefficients of the formal variables in the expressions from Proposition 3.1 we have the following sums which, when applied to any vector $v \in V_L^T$, yield zero.

Corollary 3.1. *We have*

$$(3.15) \quad R(\alpha, \beta|t) = \sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z} \\ m_1 + m_2 = -t}} x_{\alpha}^{\hat{\nu}}(m_1) x_{\beta}^{\hat{\nu}}(m_2) \text{ when } \langle \nu^j \alpha, \beta \rangle \geq 0 \text{ for } j = 0, 1,$$

$$(3.16) \quad R(\alpha, \beta|t) = \sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z} \\ m_1 + m_2 + \frac{1}{2} = -t}} \left(x_{\alpha}^{\hat{\nu}}\left(m_1 + \frac{1}{2}\right) x_{\beta}^{\hat{\nu}}(m_2) - x_{\alpha}^{\hat{\nu}}(m_1) x_{\beta}^{\hat{\nu}}\left(m_2 + \frac{1}{2}\right) \right) \\ \text{when } \langle \alpha, \beta \rangle = -1 \text{ and } \langle \nu \alpha, \beta \rangle \geq 0,$$

$$(3.17) \quad R(\alpha, \beta|t) = \sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z} \\ m_1 + m_2 + \frac{1}{2} = -t}} \left(x_{\alpha}^{\hat{\nu}}\left(m_1 + \frac{1}{2}\right) x_{\beta}^{\hat{\nu}}(m_2) + x_{\alpha}^{\hat{\nu}}(m_1) x_{\beta}^{\hat{\nu}}\left(m_2 + \frac{1}{2}\right) \right) \\ \text{when } \langle \alpha, \beta \rangle \geq 0 \text{ and } \langle \nu \alpha, \beta \rangle = -1,$$

and

$$(3.18) \quad R(\alpha, \beta|t) = \sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z} \\ m_1 + m_2 + 1 = -t}} (x_{\alpha}^{\hat{\nu}}(m_1 + 1) x_{\beta}^{\hat{\nu}}(m_2) - x_{\alpha}^{\hat{\nu}}(m_1) x_{\beta}^{\hat{\nu}}(m_2 + 1)) \\ \text{when } \langle \nu^j \alpha, \beta \rangle = -1 \text{ for } j = 0, 1,$$

when applied to any $v \in V_L^T$ yield zero.

We define the following truncated finite sums

$$(3.19) \quad R^0(\alpha, \beta|t) = \sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha}^{\hat{\nu}}(m_1) x_{\beta}^{\hat{\nu}}(m_2) \text{ when } \langle \nu^j \alpha, \beta \rangle \geq 0 \text{ for } j = 0, 1,$$

$$(3.20) \quad R^0(\alpha, \beta|t) = \sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z} < 0 \\ m_1 + m_2 + \frac{1}{2} = -t}} \left(x_\alpha^\nu \left(m_1 + \frac{1}{2} \right) x_\beta^\nu(m_2) - x_\alpha^\nu(m_1) x_\beta^\nu \left(m_2 + \frac{1}{2} \right) \right)$$

when $\langle \alpha, \beta \rangle = -1$ and $\langle \nu\alpha, \beta \rangle \geq 0$,

$$(3.21) \quad R^0(\alpha, \beta|t) = \sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z} < 0 \\ m_1 + m_2 + \frac{1}{2} = -t}} \left(x_\alpha^\nu \left(m_1 + \frac{1}{2} \right) x_\beta^\nu(m_2) + x_\alpha^\nu(m_1) x_\beta^\nu \left(m_2 + \frac{1}{2} \right) \right)$$

when $\langle \nu\alpha, \beta \rangle \geq 0$ and $\langle \nu\alpha, \beta \rangle = -1$,

and

$$(3.22) \quad R^0(\alpha, \beta|t) = \sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z} < 0 \\ m_1 + m_2 + 1 = -t}} \left(x_\alpha^\nu(m_1 + 1) x_\beta^\nu(m_2) - x_\alpha^\nu(m_1) x_\beta^\nu(m_2 + 1) \right)$$

when $\langle \nu^j \alpha, \beta \rangle = -1$ for $j = 0, 1$,

For the following proposition, we require a certain completion of $U(\widetilde{\mathfrak{n}[\nu]}) \mathfrak{n}[\nu]_+$, which we call $U(\widetilde{\mathfrak{n}[\nu]}) \widetilde{\mathfrak{n}[\nu]}_+$. We refer the reader to [LW] and [S2] for details of the construction of this completion.

Proposition 3.2. *For $\alpha, \beta \in \Delta^+$ and $R(\alpha, \beta|t)$ described above we may write*

$$(3.23) \quad R(\alpha, \beta|t) = R^0(\alpha, \beta|t) + a$$

where $R^0(\alpha, \beta|t)$ is as above and $a \in U(\widetilde{\mathfrak{n}[\nu]}) \widetilde{\mathfrak{n}[\nu]}_+$.

Proof. It is clear we can rewrite the infinite sums (3.15) in this form because in this case the operators $x_\alpha^\nu(m)$ and $x_\beta^\nu(n)$ commute, so we any term that contains a positive mode can easily be written as an element of $U(\widetilde{\mathfrak{n}[\nu]}) \widetilde{\mathfrak{n}[\nu]}_+$.

We now focus on expressions of the form (3.16); the arguments involving (3.17) and (3.18) will be similar. We first decompose

$$\begin{aligned} R(\alpha, \beta|t) &= \sum_{m_1 \geq 0, m_2 < 0} \left(x_\alpha^\nu \left(m_1 + \frac{1}{2} \right) x_\beta^\nu(m_2) - x_\alpha^\nu(m_1) x_\beta^\nu \left(m_2 + \frac{1}{2} \right) \right) \\ &\quad + R^0(\alpha, \beta|t) + \sum_{m_1 < 0, m_2 \geq 0} \left(x_\alpha^\nu \left(m_1 + \frac{1}{2} \right) x_\beta^\nu(m_2) - x_\alpha^\nu(m_1) x_\beta^\nu \left(m_2 + \frac{1}{2} \right) \right). \end{aligned}$$

The third summand is clearly a member of $U(\widetilde{\mathfrak{n}[\nu]}) \widetilde{\mathfrak{n}[\nu]}_+$ so we will focus on the first. Using Corollary 2.1 we write

$$\begin{aligned}
 & \sum_{m_1 \geq 0, m_2 < 0} \left(x_{\alpha}^{\dot{\nu}} \left(m_1 + \frac{1}{2} \right) x_{\beta}^{\dot{\nu}} (m_2) - x_{\alpha}^{\dot{\nu}} (m_1) x_{\beta}^{\dot{\nu}} \left(m_2 + \frac{1}{2} \right) \right) \\
 &= \sum_{m_1 \geq 0, m_2 < 0} \left(x_{\beta}^{\dot{\nu}} (m_2) x_{\alpha}^{\dot{\nu}} \left(m_1 + \frac{1}{2} \right) - \frac{1}{2} \epsilon_{C_0}(\alpha, \beta) x_{\alpha+\beta}^{\dot{\nu}} \left(m_1 + m_2 + \frac{1}{2} \right) \right. \\
 & \quad \left. - x_{\beta}^{\dot{\nu}} \left(m_2 + \frac{1}{2} \right) x_{\alpha}^{\dot{\nu}} (m_1) + \frac{1}{2} \epsilon_{C_0}(\alpha, \beta) x_{\alpha+\beta}^{\dot{\nu}} \left(m_1 + m_2 + \frac{1}{2} \right) \right) \\
 &= \sum_{m_1 \geq 0, m_2 < 0} \left(x_{\beta}^{\dot{\nu}} (m_2) x_{\alpha}^{\dot{\nu}} \left(m_1 + \frac{1}{2} \right) - x_{\beta}^{\dot{\nu}} \left(m_2 + \frac{1}{2} \right) x_{\alpha}^{\dot{\nu}} (m_1) \right) \in U(\widetilde{\mathfrak{n}[\dot{\nu}]}) \mathfrak{n}[\dot{\nu}]_+
 \end{aligned}$$

□

Define the following left ideals of $U(\mathfrak{n}[\dot{\nu}])$:

$$(3.24) \quad J = \sum_{\substack{t \in \mathbb{Z}_{>0} \\ \nu \alpha_i = \alpha_i}} U(\mathfrak{n}[\dot{\nu}]) R^0(\alpha_i, \alpha_i | t) + \sum_{\substack{t \in \frac{1}{2}\mathbb{Z}_{>0} \\ \nu \alpha_i \neq \alpha_i}} U(\mathfrak{n}[\dot{\nu}]) R^0(\alpha_i, \alpha_i | t)$$

as well as

$$(3.25) \quad I_{\Lambda} = J + U(\mathfrak{n}[\dot{\nu}]) \mathfrak{n}[\dot{\nu}]_+.$$

For A_{2n-1} , we let $i = 1, \dots, n$. For D_n , we let $i = 1, \dots, n-1$, and for E_6 we let $i = 1, 2, 3, 6$.

Remark 3.1. The ideal I_{Λ} is defined analogously to the ideals found in [CalLM1]-[CalLM4], [CalMPe], and [PS]. In the following sections, we investigate the properties of I_{Λ} , and, as in [CalLM1]-[CalLM4], [CalMPe], and [PS], show that it gives the desired presentation of the principal subspace.

4. IMPORTANT MAPPINGS

Following [CalLM4], [CalMPe], and [PS], we make use of certain shift automorphisms of $U(\mathfrak{n}[\dot{\nu}])$. For $\gamma \in \mathfrak{h}_{(0)}$ and a character of the root lattice $\theta : L \rightarrow \mathbb{C}$ consider the linear map

$$(4.1) \quad \begin{aligned} \tau_{\gamma, \theta} : \mathfrak{n}[\dot{\nu}] &\rightarrow \mathfrak{n}[\dot{\nu}] \\ x_{\alpha}^{\dot{\nu}}(m) &\mapsto \theta(\alpha) x_{\alpha}^{\dot{\nu}}(m + \langle \alpha_{(0)}, \gamma \rangle). \end{aligned}$$

For suitably chosen γ and θ , this is a Lie algebra automorphism that extends to an automorphism of $U(\mathfrak{n}[\dot{\nu}])$ which we also denote by $\tau_{\gamma, \theta}$ defined by

$$(4.2) \quad \begin{aligned} & \tau_{\gamma, \theta} \left(x_{\beta_{i_r}}^{\dot{\nu}}(m_r) \cdots x_{\beta_{i_1}}^{\dot{\nu}}(m_1) \right) \\ &= \theta(\beta_{i_1} + \cdots + \beta_{i_r}) x_{\beta_{i_r}}^{\dot{\nu}}(m_r + \langle (\beta_{i_r})_{(0)}, \gamma \rangle) \cdots x_{\beta_{i_1}}^{\dot{\nu}}(m_1 + \langle (\beta_{i_1})_{(0)}, \gamma \rangle), \end{aligned}$$

where $\beta_{i_j} \in \Delta^+$. As in [CalLM4], [CalMPe], and [PS], we use automorphisms associated to the elements of $\mathfrak{h}_{(0)}$ associated to the duals of the simple roots

$$(4.3) \quad \gamma_i = (\lambda_i)_{(0)} = \frac{1}{2} (\lambda_i + \nu \lambda_i)$$

where, for A_{2n-1} , we let $i = 1, \dots, n$, for D_n , we let $i = 1, \dots, n-1$, and for E_6 we let $i = 1, 2, 3, 6$. as well as the characters θ_i defined by

$$(4.4) \quad \theta_i(\alpha_j) = (-1)^{\langle \lambda^{(i)}, \alpha_j \rangle}.$$

In particular, if L is of type A_{2n-1} , we have:

$$(4.5) \quad \theta_i(\alpha_i) = -1$$

$$(4.6) \quad \theta_i(\alpha_j) = 1 \text{ if } i \neq j \text{ and } j \in \{1, \dots, 2n-1\}$$

for $i \in \{1, \dots, n-1\}$ and we have

$$(4.7) \quad \theta_n(\alpha_j) = 1 \text{ if } j \in \{1, \dots, 2n-1\}$$

If L is of type D_n , we have:

$$(4.8) \quad \theta_{n-1}(\alpha_{n-1}) = -1$$

$$(4.9) \quad \theta_{n-1}(\alpha_j) = 1 \text{ if } j \in \{1, 2, \dots, n-2, n\}$$

and define

$$(4.10) \quad \theta_i(\alpha_j) = 1 \text{ if } j \in \{1, \dots, n\}$$

for $i \in \{1, \dots, n-1\}$.

If L is of type E_6 , we have:

$$(4.11) \quad \theta_i(\alpha_i) = -1$$

$$(4.12) \quad \theta_i(\alpha_j) = 1 \text{ if } i \neq j \text{ and } j \in \{1, \dots, 6\}$$

for $i \in \{1, 2\}$ and we have

$$(4.13) \quad \theta_i(\alpha_j) = 1 \text{ if } j \in \{1, \dots, 2n-1\} \text{ and } i \in \{3, 6\}.$$

The maps $\tau_{\gamma_i, \theta_i}$ have inverses given by $\tau_{-\gamma_i, \theta_i^{-1}}$. The following Lemma follows immediately from the action of the maps $\tau_{\gamma_i, \theta_i}$:

Lemma 4.1. *We have that*

$$(4.14) \quad \tau_{\gamma_i, \theta_i} \left(U(\bar{\mathbf{n}}[\hat{\nu}]) \bar{\mathbf{n}}[\hat{\nu}]_+ + U(\bar{\mathbf{n}}[\hat{\nu}]) x_{\alpha_i}^{\hat{\nu}}(-1) \right) \subset U(\bar{\mathbf{n}}[\hat{\nu}]) \bar{\mathbf{n}}[\hat{\nu}]_+$$

if $\nu\alpha_i = \alpha_i$ and

$$(4.15) \quad \tau_{\gamma_i, \theta_i} \left(U(\bar{\mathbf{n}}[\hat{\nu}]) \bar{\mathbf{n}}[\hat{\nu}]_+ + U(\bar{\mathbf{n}}[\hat{\nu}]) x_{\alpha_i}^{\hat{\nu}} \left(-\frac{1}{2} \right) \right) \subset U(\bar{\mathbf{n}}[\hat{\nu}]) \bar{\mathbf{n}}[\hat{\nu}]_+$$

if $\nu\alpha_i \neq \alpha_i$.

Proof. This immediately follows from the fact that $\langle \gamma_i, \beta_{(0)} \rangle \geq 0$ for each $\beta \in \Delta_+$, and the fact that

$$(4.16) \quad \tau_{\gamma_i, \theta_i} \left(x_{\alpha_i}^{\hat{\nu}}(-1) \right) = \theta_i(\alpha_i) x_{\alpha_i}^{\hat{\nu}}(0) \in U(\bar{\mathbf{n}}[\hat{\nu}]) \bar{\mathbf{n}}[\hat{\nu}]_+$$

if $\nu\alpha_i = \alpha_i$ and

$$(4.17) \quad \tau_{\gamma_i, \theta_i} \left(x_{\alpha_i}^{\hat{\nu}} \left(-\frac{1}{2} \right) \right) = \theta_i(\alpha_i) x_{\alpha_i}^{\hat{\nu}}(0) \in U(\bar{\mathbf{n}}[\hat{\nu}]) \bar{\mathbf{n}}[\hat{\nu}]_+$$

if $\nu\alpha_i \neq \alpha_i$. □

We now present three Lemmas that are essential in determining $\text{Ker} f_\Lambda$. The lemmas and their proofs largely mirror similar lemmas in [CalMPe] and [PS].

Lemma 4.2. *We have*

$$(4.18) \quad \tau_{\gamma_i, \theta_i} (I_\Lambda + U(\bar{\mathfrak{n}}[\hat{\nu}]) x_{\alpha_i}^{\hat{\nu}}(-1)) = I_\Lambda$$

if $\nu\alpha_i = \alpha_i$ and

$$(4.19) \quad \tau_{\gamma_i, \theta_i} \left(I_\Lambda + U(\bar{\mathfrak{n}}[\hat{\nu}]) x_{\alpha_i}^{\hat{\nu}} \left(-\frac{1}{2} \right) \right) = I_\Lambda$$

if $\nu\alpha_i \neq \alpha_i$.

Proof. We first show that

$$(4.20) \quad \tau_{\gamma_i, \theta_i} (I_\Lambda + U(\bar{\mathfrak{n}}[\hat{\nu}]) x_{\alpha_i}^{\hat{\nu}}(-1)) \subset I_\Lambda,$$

if $\nu\alpha_i = \alpha_i$ and

$$(4.21) \quad \tau_{\gamma_i, \theta_i} \left(I_\Lambda + U(\bar{\mathfrak{n}}[\hat{\nu}]) x_{\alpha_i}^{\hat{\nu}} \left(-\frac{1}{2} \right) \right) \subset I_\Lambda.$$

if $\nu\alpha_i \neq \alpha_i$. In view of Lemma 4.1, it is sufficient to show that $\tau_{\gamma_i, \theta_i}(R^0(\alpha_j, \alpha_j|t)) \in I_\Lambda$. If $i \neq j$, we have that

$$(4.22) \quad \tau_{\gamma_i, \theta_i}(R^0(\alpha_j, \alpha_j|t)) = R^0(\alpha_j, \alpha_j|t) \in I_\Lambda.$$

If $\nu\alpha_i = \alpha_i$, we have that

$$\begin{aligned} \tau_{\gamma_i, \theta_i}(R^0(\alpha_i, \alpha_i|t)) &= \tau_{\gamma_i, \theta_i} \left(\sum_{\substack{m_1, m_2 \in \mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_i}^{\hat{\nu}}(m_1) x_{\alpha_i}^{\hat{\nu}}(m_2) \right) \\ &= \theta_i(2\alpha_i) \left(\sum_{\substack{m_1, m_2 \in \mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_i}^{\hat{\nu}}(m_1 + 1) x_{\alpha_i}^{\hat{\nu}}(m_2 + 1) \right) \\ &= \theta_i(2\alpha_i) (R^0(\alpha_i, \alpha_i|t-2) + 2x_{\alpha_i}^{\hat{\nu}}(-t+2) x_{\alpha_i}^{\hat{\nu}}(0)) \\ &\in J + U(\bar{\mathfrak{n}}[\hat{\nu}]) \bar{\mathfrak{n}}[\hat{\nu}]_+ = I_\Lambda. \end{aligned}$$

If $\nu\alpha_i \neq \alpha_i$, we have

$$\begin{aligned} \tau_{\gamma_i, \theta_i}(R^0(\alpha_i, \alpha_i|t)) &= \tau_{\gamma_i, \theta_i} \left(\sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_i}^{\hat{\nu}}(m_1) x_{\alpha_i}^{\hat{\nu}}(m_2) \right) \\ (4.23) \quad &= \theta_i(2\alpha_i) \left(\sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_i}^{\hat{\nu}} \left(m_1 + \frac{1}{2} \right) x_{\alpha_i}^{\hat{\nu}} \left(m_2 + \frac{1}{2} \right) \right) \\ &= \theta_i(2\alpha_i) (R^0(\alpha_i, \alpha_i|t-1) + 2x_{\alpha_i}^{\hat{\nu}}(-t+1) x_{\alpha_i}^{\hat{\nu}}(0)) \\ &\in I_\Lambda \end{aligned}$$

Since J is the left ideal generated by the $R^0(\alpha_i, \alpha_i|t)$, we have that $\tau_{\gamma_i, \theta_i}(J) \subset I_\Lambda$, establishing (4.20) and (4.21).

For the other containment, we will prove equivalent statements

$$(4.24) \quad \tau_{-\gamma_i, \theta_i^{-1}}(I_\Lambda) \subset I_\Lambda + U(\bar{\mathfrak{n}}[\hat{\nu}]) x_{\alpha_i}^{\hat{\nu}}(-1).$$

if $\nu\alpha_i = \alpha_i$ and

$$(4.25) \quad \tau_{-\gamma_i, \theta_i^{-1}}(I_\Lambda) \subset I_\Lambda + U(\bar{\mathfrak{n}}[\hat{\nu}]) x_{\alpha_i}^{\hat{\nu}} \left(-\frac{1}{2}\right).$$

if $\nu\alpha_i \neq \alpha_i$

Using

$$(4.26) \quad \tau_{-\gamma_i, \theta_i^{-1}}(x_{\alpha_j}^{\hat{\nu}}(m)) = \theta_1^{-1}(\alpha_i) x_{\alpha_j}^{\hat{\nu}}(m - \delta_{i,j})$$

if $\nu\alpha_i = \alpha_i$ and

$$(4.27) \quad \tau_{-\gamma_i, \theta_i^{-1}}(x_{\alpha_j}^{\hat{\nu}}(m)) = \theta_1^{-1}(\alpha_i) x_{\alpha_j}^{\hat{\nu}} \left(m - \frac{1}{2}\delta_{i,j}\right)$$

if $\nu\alpha_i \neq \alpha_i$, it is clear that

$$(4.28) \quad \tau_{-\gamma_i, \theta_i^{-1}}(U(\bar{\mathfrak{n}}[\hat{\nu}]) \bar{\mathfrak{n}}[\hat{\nu}]_+) \subset U(\bar{\mathfrak{n}}[\hat{\nu}]) \bar{\mathfrak{n}}[\hat{\nu}]_+ + U(\bar{\mathfrak{n}}[\hat{\nu}]) x_{\alpha_i}^{\hat{\nu}}(-1)$$

if $\nu\alpha_i = \alpha_i$ and

$$(4.29) \quad \tau_{-\gamma_i, \theta_i^{-1}}(U(\bar{\mathfrak{n}}[\hat{\nu}]) \bar{\mathfrak{n}}[\hat{\nu}]_+) \subset U(\bar{\mathfrak{n}}[\hat{\nu}]) \bar{\mathfrak{n}}[\hat{\nu}]_+ + U(\bar{\mathfrak{n}}[\hat{\nu}]) x_{\alpha_i}^{\hat{\nu}} \left(-\frac{1}{2}\right)$$

if $\nu\alpha_i \neq \alpha_i$. Furthermore, if $i \neq j$ we have that

$$(4.30) \quad \tau_{-\gamma_i, \theta_i^{-1}}(R(\alpha_j, \alpha_j|t)) = R(\alpha_j, \alpha_j|t) \in I_\Lambda.$$

Now we investigate the action of $\tau_{-\gamma_i, \theta_i^{-1}}$ on $R^0(\alpha_i, \alpha_i|t)$. If $\nu\alpha_i = \alpha_i$, we have

$$(4.31) \quad \begin{aligned} \tau_{-\gamma_i, \theta_i^{-1}}(R^0(\alpha_i, \alpha_i|t)) &= \tau_{-\gamma_i, \theta_i^{-1}} \left(\sum_{\substack{m_1, m_2 \in \mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_i}^{\hat{\nu}}(m_1) x_{\alpha_i}^{\hat{\nu}}(m_2) \right) \\ &= \theta^{-1}(2\alpha_i) \left(\sum_{\substack{m_1, m_2 \in \mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_i}^{\hat{\nu}}(m_1 - 1) x_{\alpha_i}^{\hat{\nu}}(m_2 - 1) \right) \\ &= \theta^{-1}(2\alpha_i) (R^0(\alpha_i, \alpha_i|t + 2) - 2x_{\alpha_i}^{\hat{\nu}}(-t - 1) x_{\alpha_i}^{\hat{\nu}}(-1)) \\ &\in I_\Lambda + U(\bar{\mathfrak{n}}[\hat{\nu}]) x_{\alpha_i}^{\hat{\nu}}(-1). \end{aligned}$$

If $\nu\alpha_i \neq \alpha_i$, we have

$$(4.32) \quad \begin{aligned} \tau_{-\gamma_i, \theta_i^{-1}}(R^0(\alpha_i, \alpha_i|t)) &= \tau_{-\gamma_i, \theta_i^{-1}} \left(\sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_i}^{\hat{\nu}}(m_1) x_{\alpha_i}^{\hat{\nu}}(m_2) \right) \\ &= \theta^{-1}(2\alpha_i) \left(\sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_i}^{\hat{\nu}} \left(m_1 - \frac{1}{2}\right) x_{\alpha_i}^{\hat{\nu}} \left(m_2 - \frac{1}{2}\right) \right) \\ &= \theta^{-1}(2\alpha_i) \left(R^0(\alpha_i, \alpha_i|t + 1) - 2x_{\alpha_i}^{\hat{\nu}} \left(-t - \frac{1}{2}\right) x_{\alpha_i}^{\hat{\nu}} \left(-\frac{1}{2}\right) \right) \\ &\in I_\Lambda + U(\bar{\mathfrak{n}}[\hat{\nu}]) x_{\alpha_i}^{\hat{\nu}} \left(-\frac{1}{2}\right). \end{aligned}$$

Since J is the left ideal generated by the $R(\alpha_i, \alpha_i|t)$, we have that

$$(4.33) \quad \tau_{-\gamma_i, \theta_i^{-1}}(J) \subset I_\Lambda + U(\bar{\mathfrak{n}}[\hat{\nu}]) x_{\alpha_i}^{\hat{\nu}}(-1).$$

if $\nu\alpha_i = \alpha_i$ and

$$(4.34) \quad \tau_{-\gamma_i, \theta_i^{-1}}(J) \subset I_\Lambda + U(\bar{\mathfrak{n}}[\hat{\nu}]) x_{\alpha_i}^{\hat{\nu}}\left(-\frac{1}{2}\right).$$

if $\nu\alpha_i \neq \alpha_i$, thus completing the proof of the reverse containment. \square

As in [CalLM4], [CalMPe], and [PS] define linear maps $\psi_{\gamma_i, \theta_i}$ for $1 \leq i \leq l$ on $U(\bar{\mathfrak{n}}[\hat{\nu}])$ by:

$$(4.35) \quad \begin{aligned} \psi_{\gamma_i, \theta_i} : U(\bar{\mathfrak{n}}[\hat{\nu}]) &\rightarrow U(\bar{\mathfrak{n}}[\hat{\nu}]) \\ a &\mapsto \tau_{-\alpha_i, \theta_i^{-1}}(a) x_{\alpha_i}^{\hat{\nu}}(-1), \end{aligned}$$

if $\nu\alpha_i = \alpha_i$ and

$$(4.36) \quad \begin{aligned} \psi_{\gamma_i, \theta_i} : U(\bar{\mathfrak{n}}[\hat{\nu}]) &\rightarrow U(\bar{\mathfrak{n}}[\hat{\nu}]) \\ a &\mapsto \tau_{-\alpha_i, \theta_i^{-1}}(a) x_{\alpha_i}^{\hat{\nu}}\left(-\frac{1}{2}\right), \end{aligned}$$

if $\nu\alpha_i \neq \alpha_i$.

In the spirit of Lemma 2.2, we present another technical lemma that will be of use for the next result.

Lemma 4.3. *Suppose α_i is a simple root and $\alpha \in \Delta_+$ such that $\langle \alpha_i, \alpha \rangle = 2$. Then $\alpha = \alpha_i$.*

Proof. We begin with the A_{2n-1} setting, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ and $\alpha = \varepsilon_k - \varepsilon_l$ for $1 \leq i \leq 2n$ and $1 \leq k < l \leq 2n+1$. We now observe that

$$(4.37) \quad \langle \alpha_i, \alpha \rangle = \langle \varepsilon_i - \varepsilon_{i+1}, \varepsilon_k - \varepsilon_l \rangle = \delta_{i,k} + \delta_{i+1,l} - \delta_{i+1,k} - \delta_{i,l} = 2.$$

This implies that $i = k$ and $i+1 = l$ and thus $\alpha = \alpha_i$.

Moving on to the D_n setting we will explicitly check the cases that are not covered by (4.37). We begin with $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq n-1$ and $\alpha = \varepsilon_k + \varepsilon_l$ for $1 \leq k < l \leq n$. We see that

$$(4.38) \quad \langle \alpha_i, \alpha \rangle = \langle \varepsilon_i - \varepsilon_{i+1}, \varepsilon_k + \varepsilon_l \rangle = \delta_{i,k} + \delta_{i,l} - \delta_{i+1,k} - \delta_{i+1,l}$$

which implies that $k = i = l$ and thus $\alpha = 2\varepsilon_k \notin \Delta_+$. The remaining D_n cases are similar.

Finally we explore the E_6 setting, again using the symmetric basis as described in (2.63). This setting separates into a four cases.

- Case 1: $\alpha_i \neq \alpha_3$ and $\alpha = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ where one of the \mathbf{v}_i is $\varepsilon_k - \varepsilon_l$ for $1 \leq k < l \leq 3$. The calculations for this case are similar to that of type A_{2n-1} as described above and will be omitted.
- Case 2: $\alpha_i \neq \alpha_3$ and $\alpha = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is so that the $\mathbf{v}_i \in \{-\frac{2}{3}\varepsilon_1 + \frac{1}{3}\varepsilon_2 + \frac{1}{3}\varepsilon_3, \frac{1}{3}\varepsilon_1 - \frac{2}{3}\varepsilon_2 + \frac{1}{3}\varepsilon_3, \frac{1}{3}\varepsilon_1 + \frac{1}{3}\varepsilon_2 - \frac{2}{3}\varepsilon_3\}$. It is clear that no combination of α_i and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ will produce $\langle \alpha_i, \alpha \rangle = 2$ in this case.
- Case 3: We have the simple root α_3 and $\alpha = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ where one of the \mathbf{v}_i is $\varepsilon_k - \varepsilon_l$ for $1 \leq k < l \leq 3$. This case is essentially the same as case 2.
- Case 4: We have the simple root α_3 and $\alpha = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is so that the $\mathbf{v}_i \in \{-\frac{2}{3}\varepsilon_1 + \frac{1}{3}\varepsilon_2 + \frac{1}{3}\varepsilon_3, \frac{1}{3}\varepsilon_1 - \frac{2}{3}\varepsilon_2 + \frac{1}{3}\varepsilon_3, \frac{1}{3}\varepsilon_1 + \frac{1}{3}\varepsilon_2 - \frac{2}{3}\varepsilon_3\}$. We first observe that for all vectors $\mathbf{u}, \mathbf{v} \in \{-\frac{2}{3}\varepsilon_1 + \frac{1}{3}\varepsilon_2 + \frac{1}{3}\varepsilon_3, \frac{1}{3}\varepsilon_1 - \frac{2}{3}\varepsilon_2 + \frac{1}{3}\varepsilon_3, \frac{1}{3}\varepsilon_1 + \frac{1}{3}\varepsilon_2 - \frac{2}{3}\varepsilon_3\}$ we have $\langle \mathbf{u}, \mathbf{v} \rangle = \frac{2}{3}$ if and only if $\mathbf{u} = \mathbf{v}$, otherwise $\langle \mathbf{u}, \mathbf{v} \rangle = -\frac{1}{3}$. The result follows immediately from this fact.

\square

The following lemmas will be useful in the next section when determining $\text{Ker} f_\Lambda$:

Lemma 4.4. *We have that*

$$(4.39) \quad \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} (U(\bar{\mathbf{n}}[\hat{\nu}]) \bar{\mathbf{n}}[\hat{\nu}]_+ + U(\bar{\mathbf{n}}[\hat{\nu}]) x_{\alpha_i}^{\hat{\nu}}(-1)) \subset I_\Lambda$$

if $\nu\alpha_i = \alpha_i$ and

$$(4.40) \quad \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} \left(U(\bar{\mathbf{n}}[\hat{\nu}]) \bar{\mathbf{n}}[\hat{\nu}]_+ + U(\bar{\mathbf{n}}[\hat{\nu}]) x_{\alpha_i}^{\hat{\nu}} \left(-\frac{1}{2} \right) \right) \subset I_\Lambda$$

if $\nu\alpha_i \neq \alpha_i$.

Proof. We begin with the case that α_i is fixed under the action of ν . We begin by examining $\psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i}$ on the elements of $\bar{\mathbf{n}}[\hat{\nu}]_+$. Assume that $m \geq 0$ and we now consider

$$(4.41) \quad \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} (x_{\alpha_i}^{\hat{\nu}}(m)) = x_{\alpha_i}^{\hat{\nu}}(m + \langle \alpha_{(0)}, \gamma_i \rangle - \langle \alpha_{(0)}, \alpha_i \rangle) x_{\alpha_i}^{\hat{\nu}}(-1),$$

where $\alpha \in \Delta_+$. In this case we have $\gamma_i = \lambda_i$ and make use of the isometry to write

$$(4.42) \quad \langle \alpha_{(0)}, \gamma_i \rangle - \langle \alpha_{(0)}, \alpha_i \rangle = \langle \alpha, \lambda_i \rangle - \langle \alpha, \alpha_i \rangle.$$

Observe that $\langle \alpha, \alpha_i \rangle \in \{-1, 0, 1, 2\}$. We use this fact to decompose this argument into a few cases.

We begin with the case that $\langle \alpha, \alpha_i \rangle = 2$ which by Lemma 4.3 we have that $\alpha = \alpha_i$ and thus $\langle \alpha, \lambda_i \rangle = 1$. Observe that

$$(4.43) \quad \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} (x_{\alpha_i}^{\hat{\nu}}(0)) = x_{\alpha_i}^{\hat{\nu}}(-1) x_{\alpha_i}^{\hat{\nu}}(-1) = R^0(\alpha_i, \alpha_i|2) \in I_\Lambda.$$

Now if $m \geq 1$ we have

$$(4.44) \quad \begin{aligned} \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} (x_{\alpha_i}^{\hat{\nu}}(m)) &= x_{\alpha_i}^{\hat{\nu}}(m-1) x_{\alpha_i}^{\hat{\nu}}(-1) \\ &= x_{\alpha_i}^{\hat{\nu}}(-1) x_{\alpha_i}^{\hat{\nu}}(m-1) \in U(\bar{\mathbf{n}}[\hat{\nu}]) \bar{\mathbf{n}}[\hat{\nu}]_+ \subset I_\Lambda. \end{aligned}$$

We now move on to the case when $\langle \alpha, \alpha_i \rangle = 1$ and thus $\langle \alpha, \lambda_i \rangle \geq 1$. In this case, we have that $\langle \alpha, \lambda_i \rangle = 1$ or $\langle \alpha, \lambda_i \rangle = 2$, and so

$$(4.45) \quad \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} (x_{\alpha_i}^{\hat{\nu}}(m)) = x_{\alpha_i}^{\hat{\nu}}(m-1 + \langle \alpha, \lambda_i \rangle) x_{\alpha_i}^{\hat{\nu}}(-1) = x_{\alpha_i}^{\hat{\nu}}(-1) x_{\alpha_i}^{\hat{\nu}}(m-1 + \langle \alpha, \lambda_i \rangle) \in U(\bar{\mathbf{n}}[\hat{\nu}]) \bar{\mathbf{n}}[\hat{\nu}]_+ \subset I_\Lambda.$$

When $\langle \alpha, \alpha_i \rangle = 0$, we have that $\langle \alpha, \lambda_i \rangle \geq 0$. In this case, we have

$$(4.46) \quad \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} (x_{\alpha_i}^{\hat{\nu}}(m)) = x_{\alpha_i}^{\hat{\nu}}(m + \langle \alpha, \lambda_i \rangle) x_{\alpha_i}^{\hat{\nu}}(-1) = x_{\alpha_i}^{\hat{\nu}}(-1) x_{\alpha_i}^{\hat{\nu}}(m + \langle \alpha, \lambda_i \rangle) \in U(\bar{\mathbf{n}}[\hat{\nu}]) \bar{\mathbf{n}}[\hat{\nu}]_+ \subset I_\Lambda.$$

Finally, we examine the case when $\langle \alpha, \alpha_i \rangle = -1$. In this case, we have that $\langle \alpha, \lambda_i \rangle = 0$ or $\langle \alpha, \lambda_i \rangle = 1$. In this case, we have

$$(4.47) \quad \begin{aligned} \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} (x_{\alpha_i}^{\hat{\nu}}(m)) &= x_{\alpha_i}^{\hat{\nu}}(m+1 + \langle \alpha, \lambda_i \rangle) x_{\alpha_i}^{\hat{\nu}}(-1) \\ &= x_{\alpha_i}^{\hat{\nu}}(-1) x_{\alpha_i}^{\hat{\nu}}(m+1 + \langle \alpha, \lambda_i \rangle) + c x_{\alpha+\alpha_i}^{\hat{\nu}}(m + \langle \alpha, \lambda_i \rangle) \in U(\bar{\mathbf{n}}[\hat{\nu}]) \bar{\mathbf{n}}[\hat{\nu}]_+ \subset I_\Lambda, \end{aligned}$$

where c is a nonzero constant given by Corollary 2.1. Finally, we have that

$$\begin{aligned} \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} (x_{\alpha_i}^{\hat{\nu}}(-1)) &= x_{\alpha_i}^{\hat{\nu}}(-2) x_{\alpha_i}^{\hat{\nu}}(-1) \\ &= 2R^0(\alpha_i, \alpha_i|3), \end{aligned}$$

so that $\psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} (U(\bar{\mathbf{n}}[\hat{\nu}]) x_{\alpha_i}^{\hat{\nu}}(-1)) \subset I_\Lambda$.

Now, we examine the case where α_i is not fixed by ν . Throughout our calculations for this case we make use of the fact that by use of the isometry ν we may write

$$(4.48) \quad \langle \alpha_{(0)}, \gamma_i \rangle - \langle \alpha_{(0)}, \alpha_i \rangle = \frac{1}{2} (\langle \alpha, \lambda_i \rangle - \langle \alpha, \alpha_i \rangle) + \frac{1}{2} (\langle \nu\alpha, \lambda_i \rangle - \langle \nu\alpha, \alpha_i \rangle)$$

Following our previous argument we use the fact $\langle \alpha, \alpha_i \rangle, \langle \nu\alpha, \alpha_i \rangle \in \{-1, 0, 1, 2\}$ to decompose into a few subcases. Those subcases without explicit calculations follow by use of the isometry ν .

First, if $\langle \alpha, \alpha_i \rangle = 2$ by Lemma 4.3 we have $\alpha = \alpha_i$ and thus $\langle \alpha, \lambda_i \rangle = 1$, $\langle \nu\alpha, \alpha_i \rangle = 0$, and $\langle \nu\alpha, \lambda_i \rangle = 0$. So we have

$$(4.49) \quad \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} (x_{\alpha_i}^{\hat{\nu}}(0)) = x_{\alpha_i}^{\hat{\nu}} \left(\frac{1}{2} \right) x_{\alpha_i}^{\hat{\nu}} \left(\frac{1}{2} \right) = R^0(\alpha_i, \alpha_i | 1) \in J \subset I_\Lambda.$$

Further if $m \geq \frac{1}{2}$ we see that

$$(4.50) \quad \begin{aligned} \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} (x_{\alpha_i}^{\hat{\nu}}(m)) &= x_{\alpha_i}^{\hat{\nu}} \left(m + \frac{1}{2} \right) x_{\alpha_i}^{\hat{\nu}} \left(\frac{1}{2} \right) \\ &= x_{\alpha_i}^{\hat{\nu}} \left(\frac{1}{2} \right) x_{\alpha_i}^{\hat{\nu}} \left(m + \frac{1}{2} \right) \in U(\bar{\mathfrak{n}}[\hat{\nu}]) \bar{\mathfrak{n}}[\hat{\nu}]_+ \subset I_\Lambda \end{aligned}$$

Now we consider the case when $\langle \alpha, \alpha_i \rangle, \langle \nu\alpha, \alpha_i \rangle \in \{0, 1\}$ thus $\langle \alpha, \lambda_i \rangle \geq \langle \alpha, \alpha_i \rangle$ and $\langle \nu\alpha, \lambda_i \rangle \geq \langle \nu\alpha, \alpha_i \rangle$. So using (4.48) we have

$$(4.51) \quad \langle \alpha_{(0)}, \gamma_i \rangle - \langle \alpha_{(0)}, \alpha_i \rangle \geq 0$$

and thus for all $m \geq 0$

$$(4.52) \quad \begin{aligned} \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} (x_{\alpha}^{\hat{\nu}}(m)) &= x_{\alpha}^{\hat{\nu}} (m + \langle \alpha_{(0)}, \gamma_i \rangle - \langle \alpha_{(0)}, \alpha_i \rangle) x_{\alpha_i}^{\hat{\nu}} \left(\frac{1}{2} \right) \\ &= x_{\alpha_i}^{\hat{\nu}} \left(\frac{1}{2} \right) x_{\alpha}^{\hat{\nu}} (m + \langle \alpha_{(0)}, \gamma_i \rangle - \langle \alpha_{(0)}, \alpha_i \rangle) \in U(\bar{\mathfrak{n}}[\hat{\nu}]) \bar{\mathfrak{n}}[\hat{\nu}]_+ \subset I_\Lambda \end{aligned}$$

Now we move on to the subcase where $\langle \alpha, \alpha_i \rangle \in \{0, 1\}$ (thus $\langle \alpha, \lambda_i \rangle \geq \langle \alpha, \alpha_i \rangle$) and $\langle \nu\alpha, \alpha_i \rangle = -1$ (thus $\langle \nu\alpha, \lambda_i \rangle \in \{0, 1\}$). Observe that in this setting we may write

$$(4.53) \quad \langle \alpha_{(0)}, \gamma_i \rangle - \langle \alpha_{(0)}, \alpha_i \rangle = \frac{1}{2} + a$$

with $a \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. Using Corollary 2.1 we may write, for $m \geq 0$,

$$(4.54) \quad \begin{aligned} \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} (x_{\alpha}^{\hat{\nu}}(m)) &= x_{\alpha}^{\hat{\nu}} \left(m + \frac{1}{2} + a \right) x_{\alpha_i}^{\hat{\nu}} \left(-\frac{1}{2} \right) \\ &= x_{\alpha_i}^{\hat{\nu}} \left(-\frac{1}{2} \right) x_{\alpha}^{\hat{\nu}} \left(m + \frac{1}{2} + a \right) + c x_{\alpha_i + \nu\alpha}^{\hat{\nu}} (m + a) \in U(\bar{\mathfrak{n}}[\hat{\nu}]) \bar{\mathfrak{n}}[\hat{\nu}]_+ \subset I_\Lambda, \end{aligned}$$

where c is a nonzero constant.

After combining cases by use of the isometry ν we are left with the final case of $\langle \alpha, \alpha_i \rangle = \langle \nu\alpha, \alpha_i \rangle = -1$, which by Lemma 2.2 we have $\nu\alpha = \alpha$ and thus $\langle \alpha, \lambda_i \rangle = \langle \nu\alpha, \lambda_i \rangle \geq 0$. This implies

$$(4.55) \quad \langle \alpha_{(0)}, \gamma_i \rangle - \langle \alpha_{(0)}, \alpha_i \rangle = 1 + a$$

where $a \in \mathbb{Z}_{\geq 0}$. Finally using Corollary 2.1, for $m \geq 0$ we have

$$(4.56) \quad \begin{aligned} \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} (x_{\alpha}^{\hat{\nu}}(m)) &= x_{\alpha}^{\hat{\nu}} (m + 1 + a) x_{\alpha_i}^{\hat{\nu}} \left(-\frac{1}{2} \right) \\ &= x_{\alpha_i}^{\hat{\nu}} \left(-\frac{1}{2} \right) x_{\alpha}^{\hat{\nu}} (m + 1 + a) + c x_{\alpha_i + \nu\alpha}^{\hat{\nu}} \left(m + \frac{1}{2} + a \right) \in U(\bar{\mathfrak{n}}[\hat{\nu}]) \bar{\mathfrak{n}}[\hat{\nu}]_+ \subset I_\Lambda, \end{aligned}$$

where c is a nonzero constant. Finally, we have that

$$\begin{aligned} \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} (x_{\alpha_i}^{\hat{\nu}} \left(-\frac{1}{2} \right)) &= x_{\alpha_i}^{\hat{\nu}} (-1) x_{\alpha_i}^{\hat{\nu}} \left(-\frac{1}{2} \right) \\ &= 2R^0(\alpha_i, \alpha_i | \frac{3}{2}), \end{aligned}$$

so that $\psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} (U(\bar{\mathfrak{n}}[\hat{\nu}]) x_{\alpha_i}^{\hat{\nu}}(-\frac{1}{2})) \subset I_\Lambda$. □

Lemma 4.5. *We have*

$$(4.57) \quad \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} (I_\Lambda + U(\bar{\mathfrak{n}}[\hat{\nu}]) x_{\alpha_i}^{\hat{\nu}}(-1)) \subset I_\Lambda,$$

if $\nu\alpha_i = \alpha_i$ and

$$(4.58) \quad \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} (I_\Lambda + U(\bar{\mathfrak{n}}[\hat{\nu}]) x_{\alpha_i}^{\hat{\nu}}(-\frac{1}{2})) \subset I_\Lambda,$$

if $\nu\alpha_i \neq \alpha_i$.

Proof. In view of Lemma 4.4, it suffices to show that

$$(4.59) \quad \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} (J) \subset I_\Lambda.$$

As with Lemma 4.1, we begin with the case that $\nu\alpha_i = \alpha_i$. First, applying $\psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i}$ to $R(\alpha_i, \alpha_i|t)$, we have that:

$$\begin{aligned} \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} (R^0(\alpha_i, \alpha_i|t)) &= \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} \left(\sum_{\substack{m_1, m_2 \in \mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_i}^{\hat{\nu}}(m_1) x_{\alpha_i}^{\hat{\nu}}(m_2) \right) \\ &= \sum_{\substack{m_1, m_2 \in \mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_i}^{\hat{\nu}}(m_1 - 1) x_{\alpha_i}^{\hat{\nu}}(m_2 - 1) x_{\alpha_i}^{\hat{\nu}}(-1) \\ &= x_{\alpha_i}^{\hat{\nu}}(-1) R^0(\alpha_i, \alpha_i|t + 2) + a R^0(\alpha_i, \alpha_i|2) \in I_\Lambda \end{aligned}$$

for some $a \in U(\bar{\mathfrak{n}}[\hat{\nu}])$. Now, we apply $\psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i}$ to $R(\alpha_j, \alpha_j|t)$, where $i \neq j$. If $\langle \alpha_i, \alpha_j \rangle = 0$, we have that

$$(4.60) \quad \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} (R^0(\alpha_j, \alpha_j|t)) = x_{\alpha_i}^{\hat{\nu}}(-1) R^0(\alpha_j, \alpha_j|t) \in I_\Lambda.$$

If $\langle \alpha_i, \alpha_j \rangle = -1$, we have two cases to consider: the case where $\nu \alpha_j = \alpha_j$ and the case where $\nu \alpha_j \neq \alpha_j$.
 If $\nu \alpha_j = \alpha_j$, we have

$$\begin{aligned}
 \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} (R^0(\alpha_j, \alpha_j | t)) &= \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} \left(\sum_{\substack{m_1, m_2 \in \mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_j}^{\hat{\nu}}(m_1) x_{\alpha_j}^{\hat{\nu}}(m_2) \right) \\
 &= \left(\sum_{\substack{m_1, m_2 \in \mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_j}^{\hat{\nu}}(m_1 + 1) x_{\alpha_j}^{\hat{\nu}}(m_2 + 1) \right) x_{\alpha_i}^{\hat{\nu}}(-1) \\
 &= x_{\alpha_i}^{\hat{\nu}}(-1) \left(\sum_{\substack{m_1, m_2 \in \mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_j}^{\hat{\nu}}(m_1 + 1) x_{\alpha_j}^{\hat{\nu}}(m_2 + 1) \right) \\
 &\quad + \left(\sum_{\substack{m_1, m_2 \in \mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_j}^{\hat{\nu}}(m_1 + 1) x_{\alpha_i + \alpha_j}^{\hat{\nu}}(m_2) \right) \epsilon_{C_0}(\alpha_j, \alpha_i) \\
 &\quad + \left(\sum_{\substack{m_1, m_2 \in \mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_i + \alpha_j}^{\hat{\nu}}(m_1) x_{\alpha_j}^{\hat{\nu}}(m_2 + 1) \right) \epsilon_{C_0}(\alpha_j, \alpha_i) \\
 &= x_{\alpha_i}^{\hat{\nu}}(-1) R^0(\alpha_j, \alpha_j | t) + [R^0(\alpha_j, \alpha_j | t - 1), x_{\alpha_i}^{\hat{\nu}}(0)] + b \in I_\Lambda
 \end{aligned}$$

for some $b \in U(\bar{\mathfrak{n}}[\hat{\nu}])\bar{\mathfrak{n}}[\hat{\nu}]_+$. If $\nu\alpha_j \neq \alpha_j$, we have

$$(4.61) \quad \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} (R^0(\alpha_j, \alpha_j|t)) = \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} \left(\sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_j}^{\hat{\nu}}(m_1) x_{\alpha_j}^{\hat{\nu}}(m_2) \right)$$

$$(4.62) \quad = \left(\sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_j}^{\hat{\nu}}(m_1 + 1) x_{\alpha_j}^{\hat{\nu}}(m_2 + 1) \right) x_{\alpha_i}^{\hat{\nu}}(-1)$$

$$(4.63) \quad = x_{\alpha_i}^{\hat{\nu}}(-1) \left(\sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_j}^{\hat{\nu}}(m_1 + 1) x_{\alpha_j}^{\hat{\nu}}(m_2 + 1) \right)$$

$$(4.64) \quad -\epsilon_{C_0}(\alpha_j, \alpha_i) \left(\sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_j}^{\hat{\nu}}(m_1 + 1) x_{\alpha_i + \alpha_j}^{\hat{\nu}}(m_2) \right)$$

$$(4.65) \quad -\epsilon_{C_0}(\alpha_j, \alpha_i) \left(\sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_i + \alpha_j}^{\hat{\nu}}(m_1) x_{\alpha_j}^{\hat{\nu}}(m_2 + 1) \right)$$

$$(4.66) \quad = x_{\alpha_i}^{\hat{\nu}}(-1) R^0(\alpha_j, \alpha_j|t - 2)$$

$$(4.67) \quad -\epsilon_{C_0}(\alpha_j, \alpha_i) \left(\sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_j}^{\hat{\nu}}(m_1 + 1) x_{\alpha_i + \alpha_j}^{\hat{\nu}}(m_2) \right)$$

$$(4.68) \quad -\epsilon_{C_0}(\alpha_j, \alpha_i) \left(\sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_i + \alpha_j}^{\hat{\nu}}(m_1) x_{\alpha_j}^{\hat{\nu}}(m_2 + 1) \right)$$

where $c \in \mathbb{C}$. We now analyze the last two sums (4.67) and (4.68). First, for (4.67), we have

$$\begin{aligned} \left(\sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_j}^{\hat{\nu}}(m_1) x_{\alpha_i + \alpha_j}^{\hat{\nu}}(m_2) \right) &= \left(\sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t+1}} x_{\alpha_j}^{\hat{\nu}}(m_1) x_{\alpha_i + \alpha_j}^{\hat{\nu}}(m_2) \right) \\ &\quad + x_{\alpha_j}^{\hat{\nu}}\left(\frac{1}{2}\right) x_{\alpha_i + \alpha_j}^{\hat{\nu}}\left(-t + \frac{1}{2}\right) + x_{\alpha_j}^{\hat{\nu}}(0) x_{\alpha_i + \alpha_j}^{\hat{\nu}}(-t + 1) \end{aligned}$$

Notice that if $\langle \nu\alpha_j, \alpha_i + \alpha_j \rangle \geq 0$, we have

$$\begin{aligned} x_{\alpha_j}^{\hat{\nu}}\left(\frac{1}{2}\right) x_{\alpha_i + \alpha_j}^{\hat{\nu}}\left(-t + \frac{1}{2}\right) + x_{\alpha_j}^{\hat{\nu}}(0) x_{\alpha_i + \alpha_j}^{\hat{\nu}}(-t + 1) &= x_{\alpha_i + \alpha_j}^{\hat{\nu}}\left(-t + \frac{1}{2}\right) x_{\alpha_j}^{\hat{\nu}}\left(\frac{1}{2}\right) + x_{\alpha_i + \alpha_j}^{\hat{\nu}}(-t + 1) x_{\alpha_j}^{\hat{\nu}}(0) \\ &\in U(\bar{\mathfrak{n}}[\hat{\nu}])\bar{\mathfrak{n}}[\hat{\nu}]_+ \end{aligned}$$

if $\langle \nu\alpha_j, \alpha_i + \alpha_j \rangle = -1$, we have

$$\begin{aligned}
 & x_{\alpha_j}^{\hat{\nu}} \left(\frac{1}{2} \right) x_{\alpha_i + \alpha_j}^{\hat{\nu}} \left(-t + \frac{1}{2} \right) + x_{\alpha_j}^{\hat{\nu}} (0) x_{\alpha_i + \alpha_j}^{\hat{\nu}} (-t + 1) \\
 &= -x_{\nu\alpha_j}^{\hat{\nu}} \left(\frac{1}{2} \right) x_{\alpha_i + \alpha_j}^{\hat{\nu}} \left(-t + \frac{1}{2} \right) + x_{\nu\alpha_j}^{\hat{\nu}} (0) x_{\alpha_i + \alpha_j}^{\hat{\nu}} (-t + 1) \\
 &= -\epsilon_{C_0}(\nu\alpha_j, \alpha_i + \alpha_j) x_{\nu\alpha_j + \alpha_i + \alpha_j}^{\hat{\nu}} (-t + 1) + x_{\alpha_i + \alpha_j}^{\hat{\nu}} \left(-t + \frac{1}{2} \right) x_{\nu\alpha_j}^{\hat{\nu}} \left(\frac{1}{2} \right) \\
 &\quad + \epsilon_{C_0}(\nu\alpha_j, \alpha_i + \alpha_j) x_{\nu\alpha_j + \alpha_i + \alpha_j}^{\hat{\nu}} (-t + 1) + x_{\alpha_i + \alpha_j}^{\hat{\nu}} (-t + 1) x_{\nu\alpha_j}^{\hat{\nu}} (0) \\
 &\in U(\bar{\mathfrak{n}}[\hat{\nu}]) \bar{\mathfrak{n}}[\hat{\nu}]_+
 \end{aligned}$$

so that (4.67) can be expressed as

$$(4.69) \quad \epsilon_{C_0}(\alpha_j, \alpha_i) \left(\sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t+1}} x_{\alpha_j}^{\hat{\nu}}(m_1 + 1) x_{\alpha_i + \alpha_j}^{\hat{\nu}}(m_2) \right) + b$$

for some $b \in U(\bar{\mathfrak{n}}[\hat{\nu}]) \bar{\mathfrak{n}}[\hat{\nu}]_+$. For the second sum (4.68), we have

$$\begin{aligned}
 & \sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_i + \alpha_j}^{\hat{\nu}}(m_1) x_{\alpha_j}^{\hat{\nu}}(m_2 + 1) \\
 &= \sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t+1}} x_{\alpha_i + \alpha_j}^{\hat{\nu}}(m_1) x_{\alpha_j}^{\hat{\nu}}(m_2) + x_{\alpha_i + \alpha_j}^{\hat{\nu}}(-t + 1) x_{\alpha_j}^{\hat{\nu}}(0) + x_{\alpha_i + \alpha_j}^{\hat{\nu}} \left(-t + \frac{1}{2} \right) x_{\alpha_j}^{\hat{\nu}} \left(\frac{1}{2} \right) \\
 &= \sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t+1}} x_{\alpha_i + \alpha_j}^{\hat{\nu}}(m_1) x_{\alpha_j}^{\hat{\nu}}(m_2) + b
 \end{aligned}$$

for some $b \in U(\bar{\mathfrak{n}}[\hat{\nu}]) \bar{\mathfrak{n}}[\hat{\nu}]_+$. Putting these results together, we see that the sums (4.67) and (4.68) can be combined and written as

$$(4.70) \quad [R(\alpha_j, \alpha_j | t - 1), x_{\alpha_i}^{\hat{\nu}}(0)] + b$$

for some $b \in U(\bar{\mathfrak{n}}[\hat{\nu}]) \bar{\mathfrak{n}}[\hat{\nu}]_+$, and so we have

$$(4.71) \quad \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} (R^0(\alpha_j, \alpha_j | t)) \in I_\Lambda$$

as desired.

We now examine the case when $\nu\alpha_i \neq \alpha_i$. Applying $\psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i}$ to $R(\alpha_i, \alpha_i | t)$, we have that:

$$\begin{aligned}
 \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} (R^0(\alpha_i, \alpha_i | t)) &= \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} \left(\sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_i}^{\hat{\nu}}(m_1) x_{\alpha_i}^{\hat{\nu}}(m_2) \right) \\
 &= \sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_i}^{\hat{\nu}} \left(m_1 - \frac{1}{2} \right) x_{\alpha_i}^{\hat{\nu}} \left(m_2 - \frac{1}{2} \right) x_{\alpha_i}^{\hat{\nu}} \left(-\frac{1}{2} \right) \\
 &= x_{\alpha_i}^{\hat{\nu}} \left(-\frac{1}{2} \right) R^0(\alpha_i, \alpha_i | t + 1) + a R^0(\alpha_i, \alpha_i | 1) \in I_\Lambda
 \end{aligned}$$

for some $a \in U(\bar{\mathfrak{n}}[\hat{\nu}])$. Now, we apply $\psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i}$ to $R(\alpha_j, \alpha_j|t)$, where $i \neq j$. If $\langle \alpha_i, \alpha_j \rangle = 0$, we have that

$$\psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} (R^0(\alpha_j, \alpha_j|t)) = x_{\alpha_i}^{\hat{\nu}} \left(-\frac{1}{2} \right) R^0(\alpha_j, \alpha_j|t) \in I_\Lambda$$

if $\langle \alpha_i, \nu \alpha_j \rangle \geq 0$ and

$$\begin{aligned} \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} (R^0(\alpha_j, \alpha_j|t)) &= \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} \left(\sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_j}^{\hat{\nu}}(m_1) x_{\alpha_j}^{\hat{\nu}}(m_2) \right) \\ &= \left(\sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_j}^{\hat{\nu}} \left(m_1 + \frac{1}{2} \right) x_{\alpha_j}^{\hat{\nu}} \left(m_2 + \frac{1}{2} \right) \right) x_{\alpha_i}^{\hat{\nu}} \left(-\frac{1}{2} \right) \\ &= - \left(\sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_j}^{\hat{\nu}} \left(m_1 + \frac{1}{2} \right) x_{\alpha_j}^{\hat{\nu}} \left(m_2 + \frac{1}{2} \right) \right) x_{\nu \alpha_i}^{\hat{\nu}} \left(-\frac{1}{2} \right) \\ &= -x_{\nu \alpha_i}^{\hat{\nu}} \left(-\frac{1}{2} \right) \left(\sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_j}^{\hat{\nu}} \left(m_1 + \frac{1}{2} \right) x_{\alpha_j}^{\hat{\nu}} \left(m_2 + \frac{1}{2} \right) \right) \\ &\quad + \frac{1}{2} \epsilon_{C_0}(\alpha_j, \nu \alpha_i) \left(\sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_j}^{\hat{\nu}} \left(m_1 + \frac{1}{2} \right) x_{\alpha_j + \nu \alpha_i}^{\hat{\nu}}(m_2) \right) \\ &\quad + \frac{1}{2} \epsilon_{C_0}(\alpha_j, \nu \alpha_i) \left(\sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\nu \alpha_i}^{\hat{\nu}}(m_1 + \frac{1}{2}) x_{\alpha_j}^{\hat{\nu}}(m_2) \right) \\ &= -x_{\nu \alpha_i}^{\hat{\nu}} \left(\frac{1}{2} \right) R^0(\alpha_j, \alpha_j|t-1) + [R^0(\alpha_j, \alpha_j|t - \frac{1}{2}), x_{\nu \alpha_i}^{\hat{\nu}}(0)] + b \in I_\Lambda \end{aligned}$$

for some $b \in U(\bar{\mathfrak{n}}[\hat{\nu}]) \bar{\mathfrak{n}}[\hat{\nu}]_+$ if $\langle \alpha_i, \nu \alpha_j \rangle = -1$ (or equivalently $\langle \nu \alpha_i, \alpha_j \rangle = -1$).

As before, if $\langle \alpha_i, \alpha_j \rangle = -1$, we have two cases to consider: the case where $\nu \alpha_j = \alpha_j$ and the case where $\nu \alpha_j \neq \alpha_j$. If $\nu \alpha_j = \alpha_j$, we have

$$\begin{aligned}
 \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} (R^0(\alpha_j, \alpha_j | t)) &= \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} \left(\sum_{\substack{m_1, m_2 \in \mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_j}^{\hat{\nu}}(m_1) x_{\alpha_j}^{\hat{\nu}}(m_2) \right) \\
 &= \left(\sum_{\substack{m_1, m_2 \in \mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_j}^{\hat{\nu}}(m_1 + 1) x_{\alpha_j}^{\hat{\nu}}(m_2 + 1) \right) x_{\alpha_i}^{\hat{\nu}} \left(-\frac{1}{2} \right) \\
 &= x_{\alpha_i}^{\hat{\nu}} \left(-\frac{1}{2} \right) \left(\sum_{\substack{m_1, m_2 \in \mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_j}^{\hat{\nu}}(m_1 + 1) x_{\alpha_j}^{\hat{\nu}}(m_2 + 1) \right) \\
 &\quad + \epsilon_{C_0}(\alpha_j, \alpha_i) \left(\sum_{\substack{m_1, m_2 \in \mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_j}^{\hat{\nu}}(m_1 + 1) x_{\alpha_i + \alpha_j}^{\hat{\nu}} \left(m_2 + \frac{1}{2} \right) \right) \\
 &\quad + \epsilon_{C_0}(\alpha_j, \alpha_i) \left(\sum_{\substack{m_1, m_2 \in \mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_i + \alpha_j}^{\hat{\nu}} \left(m_1 + \frac{1}{2} \right) x_{\alpha_j}^{\hat{\nu}}(m_2 + 1) \right) \\
 &= x_{\alpha_i}^{\hat{\nu}}(-1) R^0(\alpha_j, \alpha_j | t) + [R^0(\alpha_j, \alpha_j | t - 1), x_{\alpha_i}^{\hat{\nu}} \left(\frac{1}{2} \right)] + b \in I_\Lambda
 \end{aligned}$$

for some $b \in U(\bar{\mathfrak{n}}[\hat{\nu}]) \bar{\mathfrak{n}}[\hat{\nu}]_+$.

If $\nu\alpha_j \neq \alpha_j$, we have examine two cases: when $\langle \nu\alpha_i, \alpha_j \rangle = 0$ and when $\langle \nu\alpha_i, \alpha_j \rangle = -1$. If $\langle \nu\alpha_i, \alpha_j \rangle = 0$, we have

$$\begin{aligned}
\psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} (R^0(\alpha_j, \alpha_j | t)) &= \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} \left(\sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_j}^{\hat{\nu}}(m_1) x_{\alpha_j}^{\hat{\nu}}(m_2) \right) \\
&= \left(\sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_j}^{\hat{\nu}}\left(m_1 + \frac{1}{2}\right) x_{\alpha_j}^{\hat{\nu}}\left(m_2 + \frac{1}{2}\right) \right) x_{\alpha_i}^{\hat{\nu}}\left(-\frac{1}{2}\right) \\
&= x_{\alpha_i}^{\hat{\nu}}\left(-\frac{1}{2}\right) \left(\sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_j}^{\hat{\nu}}\left(m_1 + \frac{1}{2}\right) x_{\alpha_j}^{\hat{\nu}}\left(m_2 + \frac{1}{2}\right) \right) \\
&\quad + \frac{1}{2} \epsilon_{C_0}(\alpha_j, \alpha_i) \left(\sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_j}^{\hat{\nu}}\left(m_1 + \frac{1}{2}\right) x_{\alpha_i + \alpha_j}^{\hat{\nu}}(m_2) \right) \\
&\quad + \frac{1}{2} \epsilon_{C_0}(\alpha_j, \alpha_i) \left(\sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_i + \alpha_j}^{\hat{\nu}}(m_1) x_{\alpha_j}^{\hat{\nu}}\left(m_2 + \frac{1}{2}\right) \right) \\
&= x_{\alpha_i}^{\hat{\nu}}\left(-\frac{1}{2}\right) R^0(\alpha_j, \alpha_j | t) + [R^0(\alpha_j, \alpha_j | t - \frac{1}{2}), x_{\alpha_i}^{\hat{\nu}}(0)] + b \in I_\Lambda
\end{aligned}$$

for some $b \in U(\bar{\mathfrak{n}}[\hat{\nu}])\bar{\mathfrak{n}}[\hat{\nu}]_+$. If $\langle \nu\alpha_i, \alpha_j \rangle = -1$, we have

$$\begin{aligned}
 \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} (R^0(\alpha_j, \alpha_j|t)) &= \psi_{\gamma_i, \theta_i} \tau_{\gamma_i, \theta_i} \left(\sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_j}^{\hat{\nu}}(m_1) x_{\alpha_j}^{\hat{\nu}}(m_2) \right) \\
 &= \left(\sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_j}^{\hat{\nu}}(m_1 + 1) x_{\alpha_j}^{\hat{\nu}}(m_2 + 1) \right) x_{\alpha_i}^{\hat{\nu}}\left(-\frac{1}{2}\right) \\
 &= x_{\alpha_i}^{\hat{\nu}}\left(-\frac{1}{2}\right) \left(\sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_j}^{\hat{\nu}}(m_1 + 1) x_{\alpha_j}^{\hat{\nu}}(m_2 + 1) \right) \\
 &\quad + \epsilon_{C_0}(\alpha_j, \alpha_i) \left(\sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_j}^{\hat{\nu}}(m_1 + 1) x_{\alpha_i + \alpha_j}^{\hat{\nu}}\left(m_2 + \frac{1}{2}\right) \right) \\
 &\quad + \epsilon_{C_0}(\alpha_j, \alpha_i) \left(\sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z}_{<0} \\ m_1 + m_2 = -t}} x_{\alpha_i + \alpha_j}^{\hat{\nu}}\left(m_1 + \frac{1}{2}\right) x_{\alpha_j}^{\hat{\nu}}(m_2 + 1) \right) \\
 &= x_{\alpha_i}^{\hat{\nu}}\left(-\frac{1}{2}\right) R^0(\alpha_j, \alpha_j|t) + [R^0(\alpha_j, \alpha_j|t - \frac{3}{2}), x_{\alpha_i}^{\hat{\nu}}(0)] + b \in I_\Lambda
 \end{aligned}$$

for some $b \in U(\bar{\mathfrak{n}}[\hat{\nu}])\bar{\mathfrak{n}}[\hat{\nu}]_+$, completing our proof. \square

Consider the maps

$$(4.72) \quad e_{\alpha_i} : V_L^T \rightarrow V_L^T$$

and their restriction to the principal subspace $W_L^T \subset V_L^T$ where

$$\begin{aligned}
 (4.73) \quad e_{\alpha_i} \cdot 1 &= \frac{2}{\sigma(\alpha_i)} x_{\alpha_i}^{\hat{\nu}}(-1) \cdot 1 \text{ if } \nu\alpha_i = \alpha_i \\
 e_{\alpha_i} \cdot 1 &= \frac{2}{\sigma(\alpha_i)} x_{\alpha_i}^{\hat{\nu}}\left(-\frac{1}{2}\right) \cdot 1 \text{ if } \nu\alpha_i \neq \alpha_i,
 \end{aligned}$$

and

$$(4.74) \quad e_{\alpha_i} x_{\beta}^{\hat{\nu}}(n) = C(\alpha_i, \beta) x_{\beta}^{\hat{\nu}}(n - \langle \beta_{(0)}, \alpha_i \rangle) e_{\alpha_i}.$$

Recall that we have a basis dual to the \mathbb{Z} -basis with respect to $\langle \cdot, \cdot \rangle$, of L given by $\{\lambda_i | 1 \leq i \leq l\}$ as described in (2.89). For $\lambda \in \{\lambda_i | 1 \leq i \leq l\}$, we now introduce the map

$$(4.75) \quad \Delta^T(\lambda, -x) = (-1)^{\nu\lambda} x^{\lambda(0)} E^+(-\lambda, x).$$

This is a twisted version of the Δ -map in [Li2]. Recently such twisted maps have been exploited in [CalLM4], [CalMPe], and [PS]. We will make use the constant term of $\Delta^T(\lambda, -x)$ denoted by

$\Delta_c^T(\lambda, -x)$. Let us explore the action of $\Delta_c^T(\lambda, -x)$ on the operators $x_\beta^\nu(m)$ where $\beta \in \Delta_+$. This calculation will be made in two parts, the first being when $\nu\alpha_i = \alpha_i$ and thus $\nu\lambda_i = \lambda_i$:

$$\begin{aligned}
(4.76) \quad \Delta_c^T(\lambda_i, -x)(x_\beta^\nu(m) \cdot 1) &= \text{Coeff}_{x_0 x_1^{-m-1}}(\Delta^T(\lambda_i, -x)Y^\nu(e^\beta, x_1)) \\
&= \theta_i(\beta)\text{Coeff}_{x_1^{\langle\lambda_i, \beta\rangle - m - 1}}Y^\nu(e^\beta, x_1) \\
&= \theta_i(\beta)x_\beta^\nu(m + \langle\lambda_i, \beta\rangle) \cdot 1 \\
&= \tau_{\gamma_i, \theta_i}(x_\beta^\nu(m)) \cdot 1,
\end{aligned}$$

where we have used the fact that

$$(4.77) \quad E^+(-\lambda_i, x)E^-(-\beta, x_1) = \left(1 - \frac{x_1}{x}\right)^{\langle\lambda_i, \beta\rangle} E^-(-\beta, x_1)E^+(-\lambda_i, x).$$

Now for the case when $\nu\lambda_i \neq \lambda_i$:

$$\begin{aligned}
(4.78) \quad \Delta_c^T(\lambda_i, -x)(x_\beta^\nu(m) \cdot 1) &= \text{Coeff}_{x_0 x_1^{-m-1}}(\Delta^T(\lambda_i, -x)Y^\nu(e^\beta, x_1)) \\
&= \theta_i(\beta)\text{Coeff}_{x_1^{\frac{1}{2}(\langle\lambda_i, \beta\rangle + \langle\nu\lambda_i, \beta\rangle) - m - 1}}Y^\nu(e^\beta, x_1) \\
&= \theta_i(\beta)x_\beta^\nu\left(m + \frac{1}{2}(\langle\lambda_i, \beta\rangle + \langle\nu\lambda_i, \beta\rangle)\right) \cdot 1 \\
&= \tau_{\gamma_i, \theta_i}(x_\beta^\nu(m)) \cdot 1,
\end{aligned}$$

where we have used

$$(4.79) \quad E^+(-\lambda_i, x)E^-(-\beta, x_1) = \left(1 - \frac{x_1^{\frac{1}{2}}}{x^{\frac{1}{2}}}\right)^{\langle\lambda_i, \beta\rangle} \left(1 + \frac{x_1^{\frac{1}{2}}}{x^{\frac{1}{2}}}\right)^{\langle\nu\lambda_i, \beta\rangle} E^-(-\beta, x_1)E^+(-\lambda_i, x).$$

It follows that we have linear maps

$$\begin{aligned}
(4.80) \quad \Delta_c^T(\lambda_i, -x) : W_L^T &\rightarrow W_L^T \\
a \cdot 1 &\mapsto \tau_{\gamma_i, \theta_i}(a) \cdot 1,
\end{aligned}$$

where $a \in U(\bar{\mathfrak{n}}[\hat{\nu}])$.

5. THE MAIN RESULTS

Theorem 5.1. *We have*

$$(5.1) \quad \text{Ker } f_\Lambda = I_\Lambda,$$

or equivalently

$$(5.2) \quad \text{Ker } \pi_\Lambda = I_\Lambda \cdot v_\Lambda^N.$$

Proof. The inclusion

$$(5.3) \quad I_\Lambda \cdot v_\Lambda^N \subset \text{Ker } \pi_\Lambda$$

is trivial. The remainder of the proof will be for the reverse inclusion. Suppose that $a \in \text{ker } \pi_\Lambda \setminus I_\Lambda \cdot v_\Lambda^N$. We may assume that a is homogeneous with respect to all gradings and is a nonzero such element with smallest possible total charge. (Consequently, a has a nonzero charge for some $(\lambda^{(i)})_{(0)}$.) Among all elements of smallest possible total charge, we may assume that a is also of smallest $L^\nu(0)$ -weight. We first show that either

$$(5.4) \quad a \in I_\Lambda + U(\bar{\mathfrak{n}}[\hat{\nu}])x_{\alpha_i}^\nu(-1)$$

where $\nu\alpha_i = \alpha_i$, or

$$(5.5) \quad a \in I_\Lambda + U(\bar{\mathfrak{n}}[\hat{\nu}])x_{\alpha_i}^{\hat{\nu}}\left(-\frac{1}{2}\right),$$

where $\nu\alpha_i \neq \alpha_i$. Suppose not. Then by Lemma 4.2 we have that

$$(5.6) \quad \tau_{\gamma_i, \theta_i}(a) \cdot v_\Lambda^N \notin I_\Lambda \cdot v_\Lambda^N.$$

We also have that

$$(5.7) \quad a \cdot 1 = \pi_\Lambda(a \cdot v_\Lambda^N) = 0,$$

and so using (4.80), we have

$$(5.8) \quad \pi_\Lambda(\tau_{\gamma_i, \theta_i}(a) \cdot v_\Lambda^N) = \tau_{\gamma_i, \theta_i}(a) \cdot 1 = 0.$$

But, since

$$(5.9) \quad wt(\tau_{\gamma_i, \theta_i}(a)) < wt(a)$$

we have $\tau_{\gamma_i, \theta_i}(a) \in I_\Lambda$, contradicting the minimality of a , and so

$$(5.10) \quad a \in I_\Lambda + U(\bar{\mathfrak{n}}[\hat{\nu}])x_{\alpha_i}^{\hat{\nu}}(-1)$$

in the case that $\nu\alpha_i = \alpha_i$ or

$$(5.11) \quad a \in I_\Lambda + U(\bar{\mathfrak{n}}[\hat{\nu}])x_{\alpha_i}^{\hat{\nu}}\left(-\frac{1}{2}\right),$$

if $\nu\alpha_i \neq \alpha_i$. First, we suppose that α_i is fixed under ν and $a \in I_\Lambda + U(\bar{\mathfrak{n}}[\hat{\nu}])x_{\alpha_i}^{\hat{\nu}}(-1)$, so that

$$a = b + cx_{\alpha_i}^{\hat{\nu}}(-1)$$

for some $b \in I_\Lambda$ and $c \in U(\bar{\mathfrak{n}}[\hat{\nu}])$. Notice that b and $cx_{\alpha_i}^{\hat{\nu}}(-1)$ are of the same total $L^{\hat{\nu}}(0)$ weight and total charge as a . We have that

$$cx_{\alpha_i}^{\hat{\nu}}(-1) \cdot 1 = (a - b) \cdot 1 = 0,$$

so that $cx_{\alpha_i}^{\hat{\nu}}(-1) \cdot v_\Lambda^N \in \ker \pi_\Lambda \setminus I_\Lambda \cdot v_\Lambda^N$. We also have that

$$(5.12) \quad cx_{\alpha_i}^{\hat{\nu}}(-1) \cdot 1 = e_{\alpha_i}(\tau_{\alpha_i, \theta_i}(c) \cdot 1) = 0$$

so that, by the injectivity of e_{α_i} , we have that

$$(5.13) \quad \tau_{\alpha_i, \theta_i}(c) \cdot 1 = 0.$$

Since $\tau_{\alpha_i, \theta_i}(c)$ has lower total charge than a , we have that

$$(5.14) \quad \tau_{\alpha_i, \theta_i}(c) \in I_\Lambda.$$

In view of Lemma 4.2, we have that

$$(5.15) \quad \tau_{-\gamma_i, \theta_i^{-1}}(\tau_{\alpha_i, \theta_i}(c)) \in I_\Lambda + U(\bar{\mathfrak{n}}[\hat{\nu}])x_{\alpha_i}^{\hat{\nu}}(-1).$$

Finally, applying Lemma 4.5 immediately gives

$$(5.16) \quad \psi_{\gamma_i, \tau_i} \tau_{\gamma_i, \theta_i} \tau_{-\gamma_i, \theta_i^{-1}}(\tau_{\alpha_i, \theta_i}(c)) = cx_{\alpha_i}^{\hat{\nu}}(-1) \in I_\Lambda.$$

Thus, we have that

$$(5.17) \quad a = b + cx_{\alpha_i}^{\hat{\nu}}(-1) \in I_\Lambda,$$

which is a contradiction. A similar proof shows that if α_i is not fixed under ν

$$(5.18) \quad a = b + cx_{\alpha_i}^{\hat{\nu}}\left(-\frac{1}{2}\right)$$

for some $b \in I_\Lambda$ and $c \in U(\bar{\mathfrak{n}}[\hat{\nu}])$, then $a \in I_\Lambda$, thus giving a contradiction. \square

Theorem 5.2. *For i chosen from a subset of $\{1, 2, 3, \dots, l\}$ so that $(\alpha_i)_{(0)}$ creates a complete and non-repeating list, we have the following short exact sequences*

$$(5.19) \quad 0 \rightarrow W_L^T \xrightarrow{e_{\alpha_i}} W_L^T \xrightarrow{\Delta_c^T(\lambda_i, -x)} W_L^T \rightarrow 0$$

Proof. It is clear that for all chosen i e_{α_i} , is injective and $\Delta_c^T(\lambda_i, -x)$ is surjective. Now we take $w = a \cdot 1 \in \ker \Delta_c^T(\lambda_i, -x)$ for $a \in U(\bar{\mathfrak{n}}[\hat{\nu}])$. So we have

$$(5.20) \quad 0 = \Delta_c^T(\lambda_i, -x)w = \tau_{\gamma_1, \theta_1}(a) \cdot 1,$$

and thus $\tau_{\gamma_i, \theta_i}(a) \in I_\Lambda$ by Theorem 5.1. This implies that

$$(5.21) \quad a \in I_\Lambda + U(\bar{\mathfrak{n}}[\hat{\nu}]) x_{\alpha_i}^{\hat{\nu}}(-1),$$

if $\nu\alpha_i = \alpha_i$ and

$$(5.22) \quad a \in I_\Lambda + U(\bar{\mathfrak{n}}[\hat{\nu}]) x_{\alpha_i}^{\hat{\nu}}\left(-\frac{1}{2}\right),$$

if $\nu\alpha_i \neq \alpha_i$, by Lemma 4.2. So we have the condition that $w = a \cdot 1 \in \ker \Delta_c^T(\lambda_i, -x)$ is equivalent to the condition that $a \in I_\Lambda + U(\bar{\mathfrak{n}}[\hat{\nu}]) x_{\alpha_i}^{\hat{\nu}}(-1)$ (or $I_\Lambda + U(\bar{\mathfrak{n}}[\hat{\nu}]) x_{\alpha_i}^{\hat{\nu}}(-\frac{1}{2})$ as appropriate).

Now suppose $w = a \cdot 1 \in \text{Im } e_{\alpha_i}$ for $a \in U(\bar{\mathfrak{n}}[\hat{\nu}])$. So we can write $w = bx_{\alpha_i}^{\hat{\nu}}(-1) \cdot 1$ for $b \in U(\bar{\mathfrak{n}}[\hat{\nu}])$ if $\nu\alpha_i = \alpha_i$ or $w = b'x_{\alpha_i}^{\hat{\nu}}(-\frac{1}{2}) \cdot 1$ for $b' \in U(\bar{\mathfrak{n}}[\hat{\nu}])$ if $\nu\alpha_i \neq \alpha_i$ and thus $a \cdot 1 = bx_{\alpha_i}^{\hat{\nu}}(-1) \cdot 1 \in I_\Lambda + U(\bar{\mathfrak{n}}[\hat{\nu}]) x_{\alpha_i}^{\hat{\nu}}(-1)$ or $a \cdot 1 = b'x_{\alpha_i}^{\hat{\nu}}(-\frac{1}{2}) \cdot 1 \in I_\Lambda + U(\bar{\mathfrak{n}}[\hat{\nu}]) x_{\alpha_i}^{\hat{\nu}}(-\frac{1}{2})$ as appropriate. This implies that the condition that $w = a \cdot 1 \in \text{Im } e_{\alpha_i}$ is equivalent to the condition that $a \in I_\Lambda + U(\bar{\mathfrak{n}}[\hat{\nu}]) x_{\alpha_i}^{\hat{\nu}}(-1)$ (or $I_\Lambda + U(\bar{\mathfrak{n}}[\hat{\nu}]) x_{\alpha_i}^{\hat{\nu}}(-\frac{1}{2})$ as appropriate). So we have

$$(5.23) \quad w = a \cdot 1 \in \ker \Delta_c^T(\lambda_i, -x) \text{ if and only if } w = a \cdot 1 \in \text{Im } e_{\alpha_i},$$

and thus

$$(5.24) \quad \ker \Delta_c^T(\lambda_i, -x) = \text{Im } e_{\alpha_i}.$$

□

We use the multi-grading described earlier, and define the multigraded dimension of W_L^T by:

$$(5.25) \quad \chi(\mathbf{x}; q) = \text{tr}|_{W_L^T} x_1^{(\lambda^{(i_1)})_{(0)}} \dots x_k^{(\lambda^{(i_k)})_{(0)}} q^{2\hat{L}^{\hat{\nu}}(0)} \in q^{\frac{\dim \mathfrak{h}(1)}{16}} \mathbb{C}[[\mathbf{x}, q]],$$

where the $\lambda^{(i_j)}$ are described in (2.91-2.92). We use the modification

$$(5.26) \quad \chi'(\mathbf{x}; q) = q^{-\frac{\dim \mathfrak{h}(1)}{16}} \chi(\mathbf{x}; q) \in \mathbb{C}[[\mathbf{x}, q]].$$

Corollary 5.1. *Let $\{\alpha_{i_1}, \dots, \alpha_{i_k}\}$ be a complete set of representatives from the equivalence classes of the simple roots $\{\alpha_1, \dots, \alpha_l\}$ formed by orbits under ν , $\mathbf{m} = (m_1, m_2, \dots, m_k)$. We have the following short exact sequences*

$$(5.27) \quad 0 \rightarrow (W_L^T)_{(\mathbf{m}-\varepsilon_i, n-\sum_{j=1}^k 2\langle(\alpha_{i_j})_{(0)}, (\alpha_i)_{(0)}\rangle m_j+2)} \xrightarrow{e_{\alpha_i}} (W_L^T)_{(\mathbf{m}, n)} \xrightarrow{\Delta_c^T(\lambda_i, -x)} (W_L^T)_{(\mathbf{m}, n-2m_i)} \rightarrow 0,$$

if $\nu\alpha_i = \alpha_i$ and

$$(5.28) \quad 0 \rightarrow (W_L^T)_{(\mathbf{m}-\varepsilon_i, n-\sum_{j=1}^k 2\langle(\alpha_{i_j})_{(0)}, (\alpha_i)_{(0)}\rangle m_j+1)} \xrightarrow{e_{\alpha_i}} (W_L^T)_{(\mathbf{m}, n)} \xrightarrow{\Delta_c^T(\lambda_i, -x)} (W_L^T)_{(\mathbf{m}, n-m_i)} \rightarrow 0,$$

if $\nu\alpha_i \neq \alpha_i$. Moreover, we have

$$(5.29) \quad \chi'(\mathbf{x}; q) = \chi'(x_1, \dots, x_{i-1}, q^2 x_i, x_{i+1}, \dots, x_k; q) + x_i q^2 \chi'(q^{2\langle(\alpha_{i_1})_{(0)}, (\alpha_i)_{(0)}\rangle} x_1, \dots, q^{2\langle(\alpha_{i_k})_{(0)}, (\alpha_i)_{(0)}\rangle} x_k; q)$$

if $\nu\alpha_i = \alpha_i$ and

(5.30)

$$\chi'(\mathbf{x}; q) = \chi'(x_1, \dots, x_{i-1}, qx_i, x_{i+1}, \dots, x_k; q) + x_i q \chi'(q^{2\langle(\alpha_{i_1})_{(0)}(\alpha_i)_{(0)}\rangle} x_1, \dots, q^{2\langle(\alpha_{i_k})_{(0)}(\alpha_i)_{(0)}\rangle} x_k; q)$$

if $\nu\alpha_i \neq \alpha_i$.

Solving this recursion (cf. [A]) and using the notation $(a; q)_n = (1-a)(1-aq)(1-aq^2) \dots (1-aq^{n-1})$, we have:

Corollary 5.2. *We have*

$$(5.31) \quad \chi'(\mathbf{x}; q) = \sum_{\mathbf{m} \in (\mathbb{Z}_{\geq 0}^k)} \frac{q^{\frac{\mathbf{m}^t A \mathbf{m}}{2}}}{(q^{a_1}; q^{a_1})_{m_1} \dots (q^{a_k}; q^{a_k})_{m_k}} x_1^{m_1} \dots x_k^{m_k}$$

where $a_j = 2$ if $\nu\alpha_{i_j} = \alpha_{i_j}$, $a_j = 1$ if $\nu\alpha_{i_j} \neq \alpha_{i_j}$ and $A = 2(\langle(\alpha_{i_j})_{(0)}, (\alpha_{i_k})_{(0)}\rangle)$.

Explicitly, we have:

For $\mathfrak{g} = A_{2n-1}$, we have the $n \times n$ matrix

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & \dots & \dots & 0 \\ -1 & 2 & -1 & 0 & 0 & \dots & \dots & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \dots & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & 2 & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & -1 & 2 & -2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & -2 & 4 \end{bmatrix}$$

Note: This is a symmetrized Cartan matrix for the Lie algebra C_n with the last row multiplied by 2.

For $\mathfrak{g} = D_n$, we have the $(n-1) \times (n-1)$ matrix

$$A = \begin{bmatrix} 4 & -2 & 0 & 0 & 0 & \dots & \dots & 0 \\ -2 & 4 & -2 & 0 & 0 & \dots & \dots & 0 \\ 0 & -2 & 4 & -2 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \dots & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & 4 & -2 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & -2 & 4 & -2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & -2 & 2 \end{bmatrix}$$

Note: This is a symmetrized Cartan matrix for the Lie algebra B_{n-1} with the first $n-2$ rows multiplied by 2.

For $\mathfrak{g} = E_6$, we have

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 4 \end{bmatrix}$$

Note: This is a symmetrized Cartan matrix for the Lie algebra F_4 with the last 2 rows multiplied by 2.

Although there is a unique level one standard module for the algebras $A_{2n-1}^{(2)}$, $D_n^{(2)}$, and $E_6^{(2)}$ as well as associated twisted V_L modules, we will also consider certain shifted principal subspaces and their characters, in parallel with [CalLM3], [CalMPe], and [PS]. We will only include the general outline of the construction of the shifted subspaces along with their characters. For a detailed treatment see [PS].

For $\gamma \in \mathbb{Q} \otimes_{\mathbb{Z}} L$, we form an isomorphic shifted copy of $\hat{\mathfrak{g}}[\nu]$ denoted by

$$(5.32) \quad \hat{\mathfrak{g}}[\nu_\gamma].$$

Further, we have a representation of $\hat{\mathfrak{g}}[\nu_\gamma]$ on a shifted version of V_L^T denoted by $V_L^{T,\gamma}$, with vertex operators given by

$$(5.33) \quad Y^{\nu,\gamma}(\iota(e_\alpha), x) = Y^{\hat{\nu}}(\iota(e_\alpha), x) x^{\langle \alpha_{(0)}, \gamma \rangle},$$

whose highest weight vector is we denote by v_Λ^γ . Finally, we define the γ -shifted principal subspace

$$(5.34) \quad W_L^{T,\gamma} = U(\bar{\mathfrak{n}}[\nu_\gamma]) \cdot v_\Lambda^\gamma$$

where $\bar{\mathfrak{n}}[\nu_\gamma]$ is a shifted version of $\bar{\mathfrak{n}}[\nu]$.

Now we move to our particular set up. Recall $\gamma_i = (\lambda_i)_{(0)}$ and let $\gamma_{i_1}, \dots, \gamma_{i_k}$ be a complete list of such elements. For each $j \in \{i_1, \dots, i_k\}$ we consider the subspace $W_L^{\gamma_j}$ with its character $\chi_j'(\mathbf{x}; q)$ defined in parallel with (5.25) and (5.26). These shifted characters are related to $\chi'(\mathbf{x}; q)$ by

$$(5.35) \quad \chi_j'(\mathbf{x}; q) = \chi'(x_1, \dots, q^a x_j, \dots, x_k; q)$$

where $a = 2$ if $\nu\alpha_{i_j} = \alpha_{i_j}$ and $a = 1$ otherwise. This implies that

$$(5.36) \quad \chi_j'(\mathbf{x}; q) = \sum_{\mathbf{m} \in (\mathbb{Z}_{\geq 0}^k)} \frac{q^{\frac{\mathbf{m}^t A \mathbf{m}}{2} + a m_j}}{(q^{a_1}; q^{a_1})_{m_1} \cdots (q^{a_k}; q^{a_k})_{m_k}} x_1^{m_1} \cdots x_k^{m_k}$$

where $a_r = 2$ if $\nu\alpha_{i_r} = \alpha_{i_r}$, $a_r = 1$ if $\nu\alpha_{i_j} \neq \alpha_{i_j}$, $a = 2$ if $\nu\alpha_{i_j} = \alpha_{i_j}$, $a = 1$ if $\nu\alpha_{i_j} \neq \alpha_{i_j}$, and $A = 2 \langle (\alpha_{i_j})_{(0)}, (\alpha_{i_k})_{(0)} \rangle$.

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